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Dimensionally Regulated On-shell Renormalisation in QCD and QED

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Dimensionally Regulated On-shell Renormalisation in QCD and QED

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This thesis describes a technical advance in the treatment of massive fermion two-loop calculations in QED and QCD, which allows us to reduce complicated on-shell Feynman integrals to a large number of simple integrals, and one particularly complicated, but evaluable, one. The method extends the work of Chetyrkin and Tkachov to massive integrals, and is applicable to on-shell mass and wavefunction renormalisation.

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A b s t r a c t

This thesis describes a technical advance in the treatment of massive fermion two-loop calculations in QED and QCD, which allows us to reduce complicated on-shell Feynman integrals to a large number of simple integrals, and one particularly complicated, but evaluable, one. The method extends the work of Chetyrkin and Tkachov to massive integrals, and is applicable to on-shell mass and wavefunction renormalisation.

After an extensive review of the relevant areas of renormalisation, and of the rôle of quark masses in current algebra, we go on to use the extended technique to extract the fermion mass and wavefunction renormalisation constants to $O(\alpha_s^2)$, and to relate the running and pole masses to the bare mass and to each other. We find that the ratio of the running to the pole mass may be rather smaller than might be expected, which allows us to claim a perturbative source for a larger proportion of the strange quark constituent mass than has been usual before. In passing, we extract a number of two-loop renormalisation group coefficients, and find ourselves to be in agreement with other calculations.

We also find that the on-shell fermion wavefunction renormalisation constant is quite unexpectedly gauge invariant to two loops, and that it is relatively simply related to the mass renormalisation constant. We suggest that this is the result of such intricate calculations that there must be a field-theoretic explanation waiting to be uncovered. We relate our results to the effective theory of a static quark.

P r e f a c e

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T O M Y P A R E N T S

that this thesis is dedicated.

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— *Three quarks for Muster Mark!*
Sure he hasn't got much of a bark
And sure any he has it's all beside the mark.
[James Joyce, *Finnegans Wake*]

... But in the dynamic space of the living Rocket,
the double integral has a different meaning. To integrate here
is to operate on a rate of change so that time falls away: change is stilled. ...
'Meters per second' will integrate to 'meters.' The moving vehicle is
frozen, in space, to become architecture, and timeless.
It was never launched. It will never fall.
[Thomas Pynchon, *Gravity's Rainbow*]

Only connect. ...
[E M Forster, *Howards End*]

Chapter 1

Introduction

The least attractive feature of quantum field theory is that the basic, bare, theory makes very little physical sense. The principles on which it is grounded have a spartan elegance which is to some extent spoiled as soon as one calculates almost any observable, and finds it to be divergent. It is renormalisation which saves us, and allows us to claim field theory as physics, rather than mere mathematics. Renormalisation is, in essence, a delightful mathematical trick, which consists of nothing more offensive than the reordering of a perturbative series so that potentially troublesome terms cancel from a subset of expressions which we can consistently interpret as describing physical quantities; however, it always *seems* like sleight-of-hand!

Renormalisation is the subject of this thesis. Below, we describe how we have made a contribution to the set of tools available to field theorists by significantly extending the method of integration by parts, which was first used in this context by Chetyrkin and Tkachov [1] in 1981; and how we have contributed to field theory itself, by using this extended set to push to $O(\alpha_s^2)$ the calculation of on-shell massive-fermion mass and wavefunction renormalisation constants.

If, after we have renormalised the fermion masses, we are to make a physical statement about those masses, we must also consider non-perturbative sources of mass. So, after we have reviewed renormalisation and the renormalisation group in chapter 2, we describe these non-perturbative sources under the general heading of ‘current algebra’. After that, we can give swift introductions to the numerous mass parameters in QCD, and to the terminology and notation of wavefunction renormalisation, then finish off with a review of the effective field theory (EFT) in which one quark is given infinite mass. This chapter is rather large, but it is intended to be comprehensive enough that the later chapters can confine themselves to essentially new results.

The problem of renormalising the propagators can be made substantially easier by setting the masses to zero, and this method has been successfully used to perform

calculations up to five loops¹. This technique makes the integrals which arise from the Feynman diagrams quite tractable, and the present limit on the calculations is the limit of complexity, as huge numbers of diagrams must be marshalled. We will not follow this route, because we are able to deal with the analytic complexity of the six massive on-shell two-loop diagrams by using an extension of the method of integration by parts. This extension seems to have been used first in [8], where one of the calculations we describe was done by hand and seems, unfortunately, to contain some errors. In chapter 3, we describe how we developed the method in such a way that the number of integrals which have to be dealt with is drastically reduced (to only a single hard one, which has in fact since been done algebraically [9]), and how we can use recurrence relations, and computer algebra, to manipulate the six initial Feynman integrals into a host of more manageable ones.

Once we have the method in place, chapter 4 shows that it is easy to extend it to deal with fermion wavefunction renormalisation. In that chapter, we calculate the two-loop on-shell wavefunction renormalisation constant Z_2 , and find that it is gauge invariant to that order: this is both remarkable and inexplicable by us. We go on to use the expression for Z_2 we have derived, to calculate an expression for the wavefunction anomalous dimension $\tilde{\gamma}_F$ in the effective field theory of an infinite mass quark.

Finally, in chapter 5, we summarise our work.

¹See, in the case of three-loop calculations, refs [2,3,4]; for four-loops, refs [5]; and for five-loops, without fermions, refs [6,7].

Chapter 2

Review

This chapter is intended to provide all the background for chapters 3 and 4 so that these chapters may concentrate on new results. It consists of:

- a fast introduction to renormalisation, mentioning different regulation and renormalisation schemes, but concentrating on dimensional regulation, minimal subtraction, and on-shell renormalisation, as these are the schemes we principally use in this thesis;
- a description of the methods of the renormalisation group, introducing the mass and wavefunction anomalous dimensions, which we use and refer to throughout;
- a quick description of the well known method of integration by parts, first applied to this area by the authors of [1], which will fix our notation;
- a review of various topics in current algebra, using the term rather broadly, exploiting the physical ideas, rather than the detailed formalism;
- summaries of the problems which appear in mass and wavefunction renormalisation, which are more fully addressed in chapters 3 and 4 respectively.

2.1 Renormalisation

A field theory is specified by listing the fields in the theory and giving an expression for the Lagrangian, \mathcal{L} . The rules of field theory then lead us to Green's functions which describe the dynamics of the objects the fields represent [10,11,12]. Here, we are concerned only with QCD (specialising to QED when appropriate), and we interpret the Green's functions as describing the propagation of quarks and gluons (electrons and photons) and the interactions between them.

When we calculate Green's functions, however, we find that the presence of interactions inevitably leads to divergences¹ which threaten to make the theory meaningless. For

¹This is not true for string theories, which have the promise of being finite without renormalisation. See also refs [13,14], which describe an old suggestion that QED may, in fact, be a finite theory.

certain theories, these divergences may be brought under control by the process of *renormalisation*, so that we may obtain finite expressions for physical quantities. Chapter 3 is concerned with quark mass and chapter 4 with quark wavefunction renormalisation.

The divergences which appear in the Green's functions are, in the case of renormalisable theories, equivalent to divergent shifts in the parameters of the functions—the *bare parameters* appearing in the Lagrangian, such as coupling constants g_0 , or masses m_0 . If the bare parameters are not infinitesimal, then the shifted parameters, which should be the physical values we observe, become infinite (in perturbation theory). Conversely, we may demand that physical values be finite, if we allow the bare parameters to be infinitesimal, or adjusted by infinite amounts. The bare parameters are in principle unobservable, so that these infinities may be permitted, as long as they do not percolate into any relationship between observable quantities, and as long as the manipulations done using the divergent expressions are mathematically well-supported. This mathematical support is supplied by the procedures of regulation and renormalisation.

Renormalisation is simple [10,11,12,15,16]. We reformulate the theory with an extra parameter, λ , in such a way that the *regulated* theory is equivalent to the original one for some $\lambda = \lambda_0$, and is finite away from λ_0 . We calculate physical parameters as functions of the bare parameters in the new theory. Once we have enough, we invert our result and express the bare parameters, and all other quantities which depend on them, in terms of the (finite) physical parameters. If the theory is renormalisable, like both QCD and QED, we will find that all quantities of interest will now be free of divergences as we take λ to its limit λ_0 , recovering the original theory. During this regulation procedure, all expressions are well defined; and when we take λ to its limit, only the expressions for the unobservable bare parameters are divergent. In our case, the regulation parameter λ will be the spacetime dimension $D \in \mathbb{C}$, which will have the limit of $D = 4$.

Many regulation schemes are possible. All of them introduce unphysical effects—regulated theories lose Lorentz invariance, or gauge invariance, or unitarity, or causality, or they have poles at Euclidean momenta, or they exist in spacetimes with non-integer numbers of dimensions. They must lose something: if they did not, we would have constructed a finite physical theory and have no need at all for renormalisation.

For example, the simplest schemes involve cutoffs on the upper or lower limits of momentum integrals, or alterations to the photon propagator to give it a non-zero mass. These schemes are inadequate for all but the most elementary purposes, as the existence of a massive photon destroys gauge invariance immediately, and the simple-minded imposition of integration cutoffs destroys the Ward-Takahashi identities which guarantee that the renormalised theory is gauge invariant [10].

In Pauli-Villars regulation, an extra field of mass M is added to the theory. The

scheme changes the propagator $\Pi(q^2, m^2)$ for a scalar field to $\Pi_{\text{reg}}(q^2) = \Pi(q^2, m^2) - \Pi(q^2, M^2)$ for some large mass M . In the momentum integration, the second term ‘turns on’ at higher momentum, and cancels the first. Then, as $M \rightarrow \infty$, the extra part of the propagator $\Pi(q^2, M^2) \rightarrow 0$ and $\Pi_{\text{reg}} \rightarrow \Pi$. Pauli–Villars regularisation has traditionally been a popular scheme for calculations in QED [12], but although it may be used in principle for QCD, it is rarely used in practice.

The scheme which is normally used in non-Abelian theories is *dimensional regulation*, which regulates by modifying the dimension of spacetime. This scheme preserves most of the symmetries of the Lagrangian, with these exceptions: it destroys dilatation invariance, because in $D = 4 - 2\omega \neq 4$ dimensions the coupling constant acquires a mass dimension, so that a mass scale μ enters the theory; and it destroys chiral invariance because we cannot define γ_5 fully consistently in D dimensions. This is the scheme we shall use throughout this thesis, and is dealt with below, in section 2.1.1.

We should note here that in the Pauli–Villars scheme, for example, one can make a distinction between infra-red and ultra-violet divergences. The same distinction is possible in dimensional regulation of one-loop expressions, but not in two-loop expressions, where extra factors of $1/\omega$ are produced by the method of integration by parts.

Throughout the following, it should be remembered that we are interested in the physical theory which is the end product of renormalisation, rather than the manipulable regulated theory. As a result, the renormalised theory should be independent not only of the regulation parameter, but also of the choice of regulation scheme. Specifically, if we use dimensional regulation, then the renormalised theory is independent of the mass scale μ which appears, a demand which gives rise to the important *renormalisation group*, discussed below in section 2.1.5.

The material in this chapter applies to both QED and QCD, in principle. However, any calculations we do here will be done only in one-loop QED (quoting the corresponding results for QCD if they will be needed), and certain sections, for example the references to asymptotic freedom, are relevant only to QCD.

2.1.1 Dimensional regulation

We now describe the method of dimensional regulation in some detail, and show how integrals over Minkowski spacetime can be transformed into functions of the regulation parameter D . In the next section, we will show how the regulated expressions can be *renormalised* by isolating the singular dependence on D .

Consider the integral

$$I_D = -i \int_{\mathbb{M}^D} d^D k f(-k^2) \quad (2.1)$$

in arbitrary dimension D . To evaluate this, we Wick rotate the Minkowski momentum $k = (k_0, \mathbf{k}) \mapsto \kappa = (\kappa_0, \mathbf{k}) \in \mathbb{E}^D$, so that κ is a D -dimensional Euclidean vector. That is, we transform $k_0 = i\kappa_0$ so that $dk_0 = i d\kappa_0$ and $k^2 = k_0^2 - \mathbf{k}^2 = -\kappa^2$. Since κ is Euclidean, we may express it in polar coordinates $(r, \phi, \theta_1, \dots, \theta_{D-2})$ with $0 < r$, $0 \leq \phi < 2\pi$ and $0 \leq \theta_i < \pi$. This means that

$$\begin{aligned} d^D \kappa &= dr (r d\phi) (r \sin \theta_1 d\theta_1) \cdots (r \sin^{D-2} \theta_{D-2} d\theta_{D-2}) \\ &= r^{D-1} dr \cdot d^{D-1} \Omega. \end{aligned}$$

Since the integrand $f(-k^2) = f(\kappa^2)$ is independent of the solid angle Ω , we can do the Ω -integration separately:

$$\begin{aligned} \int d^{D-1} \Omega &= \int_0^{2\pi} d\phi \int_0^\pi \sin \theta_1 d\theta_1 \cdots \int_0^\pi \sin^{D-2} \theta_{D-2} d\theta_{D-2} \\ &= 2\pi \prod_{i=1}^{D-2} \int_0^\pi \sin^i \theta_i d\theta_i. \end{aligned}$$

Using the identity $\int_0^{\pi/2} d\theta \sin^{2a-1} \theta \cdot \cos^{2b-1} \theta = \frac{1}{2} B(a, b)$ with $b = \frac{1}{2}$, we find

$$\int_0^\pi d\theta \sin^i \theta = \sqrt{\pi} \frac{\Gamma((i+1)/2)}{\Gamma((i+2)/2)}$$

giving

$$\int d^{D-1} \Omega = 2 \frac{\pi^{D/2}}{\Gamma(D/2)}. \quad (2.2)$$

If we take as example the integrand $f(\kappa^2) = (\kappa^2 + x)^{-n}$, then

$$I_D = \int_{\mathbb{E}^D} d^D \kappa \frac{1}{(\kappa^2 + x)^n} = \int d^{D-1} \Omega \int_0^\infty \frac{dr r^{D-1}}{(r^2 + x)^n}.$$

With the substitution $y = r^2/x$, and using (2.2),

$$I_D = \frac{\pi^{D/2}}{\Gamma(D/2)} x^{D/2-n} \int_0^\infty \frac{y^{D/2-1}}{(1+y)^n} dy.$$

And finally, substituting $z = 1/(1+y)$ and using the definition of the beta function, we end up with

$$I_D = \pi^{D/2} x^{D/2-n} \frac{\Gamma(n - D/2)}{\Gamma(n)}, \quad n - D/2 > 0. \quad (2.3)$$

For $D \in \mathbb{Z}$, this is an unremarkable integral, which is pathological only to the extent that it is divergent when $(n - D/2)$ is a non-positive integer. We may, however, regard the quantity I_D as simply a function of a number D , conveniently forget its origin as an integration², and allow D to take on values in \mathbb{C} . Expressing D as $D = 4 - 2\omega$, and specialising to $n = 2$, we can reexpress I_D as

$$\begin{aligned} I_D &= \pi^{2-\omega} x^{-\omega} \Gamma(\omega) \\ &= \frac{\pi^2}{\omega} - \pi^2(\ln \pi x + \gamma_e) + O(\omega), \end{aligned} \quad (2.4)$$

using the properties of the gamma function outlined in section E.3.

In this way, the divergent dependence of I_D on ω is made manifest. This form suggests a tactic for the next step, that of renormalisation, and if we informally follow the prescription known as *Minimal Subtraction* (MS), we simply delete the pole in this expression before letting $\omega \rightarrow 0$. If the integral I_D were to appear in an expression for some physical quantity, then its contribution to the renormalised expression would be

$$I_D^{\text{ren}} = -\pi^2(\ln \pi x + \gamma_e). \quad (2.5)$$

2.1.2 Renormalisation: The minimal subtraction (MS) scheme

Once we have regulated the theory, we have *finite* expressions for the Green's functions, which are divergent as we take the limit of four dimensions. In renormalisation, we reparameterise the theory by systematically changing from the bare parameters initially present in the Lagrangian to new functions of these, in terms of which the Green's functions are finite. Although there is substantial arbitrariness in the choice of these functions, the choice is constrained by the demand that the rule we choose be applicable at each vertex, and should remove divergencies at all the orders of perturbation theory we are concerned with. If it can be proved that we can do this for all orders, then the theory has been proved *renormalisable*.

In this section, we will illustrate the minimal subtraction scheme by deriving an expression for the one-loop renormalised mass of the electron in QED. This also provides a convenient place to define some of our notation.

We denote by $-i\Sigma \equiv \text{---}\textcircled{\text{blob}}\text{---}$ (where --- denotes truncation of an external line) the *proper self-energy*—the sum of all one-particle-irreducible (1PI) graphs, that is, for QED:

$$\text{---}\textcircled{\text{blob}}\text{---} = \text{---}\text{cloud}\text{---} + \dots$$

²This procedure *can* be justified with some rigour. For a discussion of the analytic continuation of this expression to arbitrary D , see eg [15], sections 4.2, 3.5 and references there. The crucial point is that the above manipulations are possible when $n > D/2$, and that the analytic continuation of the Γ function is unique, so that we may take (2.3) to be the *definition* of I_D .

We denote by $\text{---}\bigcirc\text{---}$ the *complete propagator*—the sum of all connected diagrams.

Thus

$$\begin{aligned} \text{---}\bigcirc\text{---} &= \text{---}\rightarrow\text{---} + \text{---}\bigcirc\text{---} + \text{---}\bigcirc\text{---}\bigcirc\text{---} + \dots \\ &= \sum_{n=0}^{\infty} \text{---}\rightarrow\text{---}(\bigcirc\text{---})^n \\ &= \frac{\text{---}\rightarrow\text{---}}{1 - (\bigcirc\text{---})}, \end{aligned}$$

using the expression $\sum_{n=0}^{\infty} ax^n = a/(1-x)$ for the infinite sum of a geometric series. This means that, with $\rightarrow = i/(\not{p} - m_0)$, the complete propagator is

$$\begin{aligned} \text{---}\bigcirc\text{---} &= \frac{i}{\not{p} - m_0} \cdot \frac{1}{1 - (-i\Sigma) i/(\not{p} - m_0)} \\ &= \frac{i}{\not{p} - m_0 - \Sigma}. \end{aligned} \quad (2.6)$$

Thus we see that the self-energy Σ , by altering the propagator, shifts the mass of the electron.

To one-loop order, the proper self-energy is

$$-i\Sigma(\not{p})_{\text{1loop}} \equiv \text{---}\bigcirc\text{---}^{\text{1loop}} = \text{---}\rightarrow\text{---} \text{ (loop) } \text{---}\rightarrow\text{---}$$

and we may choose to write Σ in the form

$$\Sigma = m_0 \Sigma_A + \not{p} \Sigma_B, \quad (2.7)$$

so that $\text{Tr } \Sigma = 4m_0 \Sigma_A$ and $\text{Tr } \not{p} \Sigma = 4p^2 \Sigma_B$. Using the Feynman rules of appendix D.2 we find, after a standard calculation, that

$$\text{Tr } \Sigma = -4ig_0^2 m_0 C_F (D + a - 1) \frac{1}{\mu^{2\omega}} I(1, 1; p) \quad (2.8a)$$

$$\begin{aligned} \text{Tr } \not{p} \Sigma &= -2ig_0^2 C_F \frac{1}{\mu^{2\omega}} \\ &\quad \left[I(2, 1; p)(p^2 - m_0^2)^2(a - 1) \right. \\ &\quad - I(2, 0; p)2(p^2 - m_0^2)(a - 1) \\ &\quad + I(2, -1; p)(a - 1) \\ &\quad - I(1, 1; p)(p^2 + m_0^2)(D + a - 3) \\ &\quad - I(1, 0; p)(D + a - 3) \\ &\quad \left. + I(0, 1; p)(D - 2) \right], \end{aligned} \quad (2.8b)$$

where $I(\alpha, \beta; p)$ is defined in appendix E to be

$$I(\alpha, \beta; p) \equiv \mu^{2\omega} \int_{\mathbb{M}^D} \frac{d^D k}{(2\pi)^D} \frac{1}{k^{2\alpha} [(p+k)^2 - m_0^2]^\beta},$$

and μ is a quantity with the dimensions of mass. The dimension of I is thus $[I] = (\text{mass})^{4-2(\alpha+\beta)}$.

Note that $I(\alpha, \beta; p)$ is zero for β a non-positive integer, so that we are left with only $I(0, 1; p)$, $I(1, 1; p)$ and $I(2, 1; p)$ to evaluate. These specific cases are done in appendix E, so that (2.8) above reduces to

$$\begin{aligned} \text{Tr } \Sigma = \frac{4g_0^2 m_0 C_F}{(4\pi)^2 \mu^{2\omega}} & \left((a+3) \frac{1}{\omega} + \left(\ln 4\pi - \gamma_e + \ln \frac{\mu^2}{m_0^2} \right) (a+3) \right. \\ & \left. + (a+3)(m_0^2/p^2 - 1) \ln \left(1 - \frac{p^2}{m_0^2} \right) + 2(a+2) \right) \end{aligned} \quad (2.9a)$$

$$\begin{aligned} \text{Tr } \not{p} \Sigma = \frac{4g_0^2 C_F}{(4\pi)^2 \mu^{2\omega}} p^2 & \left(-\frac{a}{\omega} - \left(\ln 4\pi - \gamma_e + \ln \frac{\mu^2}{m_0^2} \right) a \right. \\ & \left. + \ln \left(1 - \frac{p^2}{m_0^2} \right) a \frac{p^4 - m_0^4}{p^4} \right. \\ & \left. - \left(\frac{m_0^2}{p^2} + 1 \right) a \right). \end{aligned} \quad (2.9b)$$

Here we see the reason for the inclusion of the factor of $\mu^{2\omega}$ in the definition of $I(\alpha, \beta; p)$. Were this not present, the term $\ln \mu^2/m_0^2$ would be $-\ln m_0^2$, the log of a dimensionful quantity. It is through this that dimensional regulation inevitably introduces a mass scale, μ , to the theory. Equivalently, the scale μ can be said to have been introduced to compensate for the mass dimension acquired by the coupling g_0 when we moved from 4 to D dimensions, ensuring that $\text{Tr } \Sigma$ and $\text{Tr } \not{p} \Sigma$ have integer mass dimensions.

$\text{Tr } \Sigma$ and $\text{Tr } \not{p} \Sigma$ together give the regulated expression for the electron self-energy Σ . This expression is divergent in the limit where $\omega \rightarrow 0$. To renormalise the theory, we must give a well-defined prescription which will allow us to consistently remove the divergent $1/\omega$ terms in the regulated expression. Once we have done this, we will have expressions for the Green's functions which are finite in the $\omega \rightarrow 0$ limit.

To provide the prescription we need, we think of the mass and coupling being multiplicatively renormalised from their bare values to finite ‘physical’ values by the interactions of the theory. This interpretation is analogous to the idea of effective masses in solid-state physics, for example. Thus, to renormalise the expressions in (2.9), we replace the bare parameters m_0 and, implicitly, ψ_0 appearing in it with their renormalised equivalents *via*

$$m_0 = Z_m m_r \quad \psi_0 = Z_2^{1/2} \psi_r \quad (2.10)$$

The renormalisation constants Z are of the form

$$Z = 1 + \frac{\alpha_0}{\pi} \frac{Z^{(1)}}{\omega} + O(\alpha_0^2)$$

where the dimensionless bare coupling α_0 is defined through

$$\alpha_0 \equiv \frac{g_0^2}{4\pi\mu^{2\omega}}.$$

The coupling α_0 and the bare gauge parameter a_0 are also renormalised. In this calculation, however, their renormalisation constants appear only in $O(\alpha^2)$ terms. Since, in this chapter, we are performing the calculation only to one loop, we are ignoring $O(\alpha^2)$ terms, so that the renormalisation of the gauge parameter and coupling constant will remain implicit, and we shall drop their subscripts for the moment. Similarly, Σ is of order α , so that we will not distinguish renormalised Σ from unrenormalised below. Also, rather than cluttering expressions by explicitly showing missing orders, in this section we will generally denote equality to $O(\alpha)$ by the symbol \simeq .

The (unrenormalised) fermion propagator is

$$\tilde{S}_F(x) = \langle 0 | T \psi_0(x) \bar{\psi}_0(0) | 0 \rangle,$$

and is the Green's function of the kinetic term in the Lagrangian D.1. This means that it satisfies

$$(i\partial - m_0)\tilde{S}_F(x) = i\delta^4(x),$$

and can therefore be verified to have the Fourier representation

$$\tilde{S}_F(x) = \int \frac{d^4p}{(2\pi)^4} e^{-ip \cdot x} \frac{i(\not{p} + m_0)}{p^2 - m_0^2 + i\epsilon}.$$

In this expression, the $\epsilon = 0_+$ defines the contour for the p integration: it, and terms like it, will be dropped in the sequel. $\tilde{S}_F(x)$ is thus the Fourier transform of $S_F(p)$, where

$$S_F(p) = \frac{i}{\not{p} - m_0},$$

which is what we will generally mean when we refer to the fermion propagator below. In the presence of interactions, $S_F(p)$ is modified such that

$$iS_F^{-1}(p) = \not{p} - m_0 - \Sigma = \not{p}(1 - \Sigma_B) - m_0(1 + \Sigma_A)$$

where Σ has been split into Σ_A and Σ_B as in (2.7). In terms of the renormalised field of eqn (2.10), we may write $S_F(p) = Z_2 S_r(p)$, defining the renormalised propagator S_r , with a similar equation for its Fourier transform. Similarly, $m_r = m_0/Z_m$ is defined so that the mass term in the bare Lagrangian, $m_0 \psi_0 \bar{\psi}_0$, is changed to $Z_m Z_2 m_r \psi_r \bar{\psi}_r$ in the renormalised Lagrangian.

Thus, we have

$$iS_F^{-1} = iZ_2^{-1} S_r^{-1} = \not{p}(1 - \Sigma_B) - m_r Z_m(1 + \Sigma_A).$$

This means that

$$iS_r^{-1} = \not{p}Z_2(1 - \Sigma_B) - m_r Z_m Z_2(1 + \Sigma_A) \quad (2.11)$$

so that Z_2 and Z_m cancel infinities in Σ_B and Σ_A . Since the $\Sigma_{A,B}$ are $O(\alpha)$ anyway, we can directly substitute $m_0 \rightarrow m_r$ into (2.9) above, and use $\Sigma_B = \text{Tr} \not{p} \Sigma / 4p^2$, to find

$$\begin{aligned} \not{p}Z_2(1 - \Sigma_B) &\simeq \not{p} \left(1 + \frac{\alpha}{\pi} \frac{Z_2^{(1)}}{\omega} \right) \\ &\times \left(1 - \frac{1}{4p^2} \frac{1}{2\mu^2\omega} \frac{\alpha}{\pi} \left[-\frac{1}{\omega} ((4 - 2\omega) + 2a - 4)p^2 + (\text{finite}) \right] \right), \end{aligned}$$

where α and a are the (renormalised) coupling and gauge parameter. *Minimal subtraction* consists of adjusting the constant Z_2 to cancel precisely these poles which we have identified in the regulated expression. Thus, to remove only the divergent part of Σ_B , we must set $Z_2^{(1)} = -a/4$. After a similar calculation for $m_r Z_m Z_2(1 + \Sigma_A)$, we find, finally, the minimally subtracted renormalisation constants

$$\begin{aligned} Z_2^{\text{MS}} &\simeq 1 - \frac{\alpha}{\pi} \frac{a}{4\omega} \\ Z_m^{\text{MS}} &\simeq 1 - \frac{\alpha}{\pi} \frac{3}{4} \frac{1}{\omega}. \end{aligned} \quad (2.12)$$

Defining the renormalised self-energy Σ_r through $iS_r^{-1} = \not{p} - m_r - \Sigma_r$ and using (2.11), we can see that

$$\begin{aligned} \Sigma_r &= \not{p}[1 - Z_2(1 - \Sigma_B)] - m_r[1 - Z_m Z_2(1 + \Sigma_A)] \\ &= \not{p}[\Sigma_B^{(0)} + O(\omega)] + m_r[\Sigma_A^{(0)} + O(\omega)] + O(\alpha^2/\pi^2) \end{aligned}$$

where $\Sigma_{A,B}^{(0)}$ are the $O(\omega^0)$ parts of the self-energy (2.7).

This expression is finite in the limit $\omega \rightarrow 0$, so that we may finally return to $D = 4$ dimensions, and give the renormalised self-energy of the electron, to one loop, as [16]

$$\begin{aligned}
\Sigma_\tau \simeq & -\not{p} \frac{\alpha}{4\pi} \left(\left(\ln 4\pi - \gamma_e + \ln \frac{\mu^2}{m^2} \right) a \right. \\
& + \ln \left(1 - \frac{p^2}{m^2} \right) \left(\frac{m^4}{p^4} - 1 \right) a \\
& \left. + a \left(\frac{m^2}{p^2} + 1 \right) \right) \\
& + m \frac{\alpha}{4\pi} \left((3+a) \left(\ln 4\pi - \gamma_e + \ln \frac{\mu^2}{m^2} \right) \right. \\
& \left. + (3+a)(m^2/p^2 - 1) \ln \left(1 - \frac{p^2}{m^2} \right) + 2(a+2) \right).
\end{aligned}$$

By comparing this expression with (2.9) (remember that $\Sigma = m_0 \Sigma_A + \not{p} \Sigma_B$, giving Σ_i through $\text{Tr } \Sigma = 4m_0 \Sigma_A$ and $\text{Tr } \not{p} \Sigma = 4p^2 \Sigma_B$), notice that in one-loop dimensional regulation, the renormalised expressions can be obtained by simply deleting the $1/\omega$ poles in the regulated expression, and immediately taking the limit as $D \rightarrow 4$.

We will need, in section 2.1.5, the expression for the one-loop vertex renormalisation constant. From, for example, ref [10], we have

$$\alpha_0 = \alpha Z_\alpha \quad (2.13)$$

$$= \alpha \left(1 + \frac{\alpha}{\pi} \frac{1}{3\omega} + O(\alpha^2) \right). \quad (2.14)$$

The terms which appear in the perturbation expansion of Σ_R are of rather large magnitude, principally because of the size of the terms $\ln 4\pi - \gamma_e \approx 1.95$. It is convenient to modify the MS scheme by the substitution

$$\mu = \bar{\mu} \left(\frac{e^{\gamma_e}}{4\pi} \right)^{1/2} \approx 0.38 \bar{\mu},$$

and so remove the combination $\gamma_e - \ln 4\pi$ from the above expression. This modified minimal subtraction scheme is known as the $\overline{\text{MS}}$ scheme.

2.1.3 Counterterms

An alternative way to see through this procedure is provided by the method of counterterms. Formally, the divergent parts can be eliminated by a procedure which regards the mass m and coupling g of the original Lagrangian as the physical ones, and removes the divergences arising from them not by a multiplicative redefinition of the mass and coupling, but by cancelling them against the interactions produced by suitably chosen *counterterms* in the Lagrangian (see, for example, [12] p326, or [15] p89). These counterterms, which have the

form of new interactions in the theory, enter with coefficients which diverge as the regulation parameter goes to its limit.

Taking as example

$$\bar{\psi}(i\partial + eA - m)\psi, \quad (2.15)$$

the kinetic term of the QED Lagrangian in eqn (D.1), we may regard ψ , m and e as the physical wavefunction, mass and coupling, add counterterms to produce

$$\begin{aligned} \mathcal{L}_{\text{kin}} = & \bar{\psi}(i\partial - m)\psi \\ & + \delta Z_2 \bar{\psi}(i\partial - m)\psi + \delta m(\delta Z_2 + 1)\bar{\psi}\psi \\ & + e'\bar{\psi}A\psi, \end{aligned}$$

where $e' = e(\delta Z_2 + 1)$, and adjust δZ_2 and δm to produce a finite result. This changes the propagator to

$$\begin{aligned} \text{---}\!\!\!\!\!\rightarrow \quad + \quad \text{---}\!\!\!\!\!\times\text{---} \quad + \quad \text{---}\!\!\!\!\!\text{wavy}\!\!\!\!\!\text{---} \\ G_0 \quad + \quad G_0[\delta Z_2(\not{p} - m) + (\delta Z_2 + 1)\delta m]G_0 \quad + \quad G_0(-i\Sigma)G_0 \end{aligned}$$

where $G_0 = i/(\not{p} - m) = \text{---}\!\!\!\!\!\rightarrow$. The self-energy Σ is recalculated with this modified propagator, and the demand that it is finite fixes δZ_2 and δm .

It should be emphasised that this is quite equivalent to the procedure in the previous section, in which we regarded the quantities in expression (2.15) as bare ones, and made the modifications

$$\begin{aligned} \psi & \rightarrow Z_2^{1/2}\psi \\ m & \rightarrow Z_m m = m - \delta m, \end{aligned}$$

where $Z_2 \equiv \delta Z_2 + 1$, and ψ and m on the right hand side are renormalised quantities.

The proof that renormalisation is a well-defined procedure can be expressed in terms of counterterms. Arguably the most natural statement that a theory is renormalisable is to say that it is so if the counterterms have the same forms as the terms which they modify in the original Lagrangian. Although it is a satisfying and elegant formalism, and has a place in any review of renormalisation, it is inconvenient for the calculations we intend to do, and we have not used it in this thesis.

2.1.4 Other renormalisation schemes

The essence of the $\overline{\text{MS}}$ and $\overline{\text{MS}}$ schemes is that the renormalisation constants are determined by demanding that they remove only the divergent part of the regulated expression, and

leave the finite part unchanged. This is not, of course, the only renormalisation scheme possible. Two popular alternatives to the $\overline{\text{MS}}$ scheme in QCD and in QED, are the μ -scheme, in which the regulated expression is renormalised by subtracting its value at an arbitrary Euclidean point $p^2 = -\mu^2$; and the Weinberg scheme (sometimes referred to as the W -scheme), a cross between the $\overline{\text{MS}}$ and μ -schemes in which the subtraction is done at a Euclidean momentum, with all quark (or electron) masses set equal to zero. Both these schemes are to some extent more physically attractive than the $\overline{\text{MS}}$ -scheme, and they have advantages in certain circumstances (the decoupling theorem, to take one example, is only at all evident in the μ -scheme), but their extra mathematical complications, in a field overburdened with algebra, make the $\overline{\text{MS}}$ scheme the most popular in QCD.

Another alternative, and one we shall also use below, is the mass-shell or physical scheme, in which a ‘physical’ mass $M = Z_M^{-1}m_0$ is defined, with Z_M determined subject to the demand that the renormalised propagator has a pole at the ‘pole mass’ M . We return to this topic in section 2.5.

2.1.5 The renormalisation group

From this section on, for the sake of clarity, we will continue to write bare parameters with the subscript 0, but write no subscript r for renormalised parameters. Specifically, $m_r(\mu)$ will be written simply as $m(\mu)$. Also, many of the expressions in this section are quoted or derived only to leading order in α : in this section again, we will not write the missing orders, and denote these cases by the approximation symbol \simeq .

We have seen how to obtain a renormalised expression for the self energy Σ . Note that, because we have used dimensional regulation, Σ depends on the mass scale μ introduced to make the coupling α dimensionless (in different renormalisation schemes, this dependence would be in terms of a different scheme parameter—we shall ignore this generality in the discussion below).

Although Σ depends on μ , its *unrenormalised* counterpart Σ_0 will not. In general, the unrenormalised, proper, n -point Green’s function $\Gamma_0^{(n)} = Z_2^{-n/2}\Gamma^{(n)}(p_i; \alpha, m, a; \mu)$, cannot depend on the renormalisation scale, so that we must have

$$\mu \frac{d}{d\mu} \Gamma_0^{(n)}(p_i; \alpha_0, a_0, m_0) = \mu \frac{d}{d\mu} \left[Z_2^{-n/2} \Gamma^{(n)}(p_i; \alpha, a, m; \mu) \right] = 0,$$

or

$$\left[\mu \frac{\partial}{\partial \mu} + \mu \frac{d\alpha}{d\mu} \frac{\partial}{\partial \alpha} + \mu \frac{dm}{d\mu} \frac{\partial}{\partial m} + \mu \frac{da}{d\mu} \frac{\partial}{\partial a} + \mu \frac{dZ_2}{d\mu} \frac{\partial}{\partial Z_2} \right] Z_2^{-n/2} \Gamma^{(n)}(p_i; \alpha, a, m; \mu) = 0.$$

We now define the quantities

$$Z_2\gamma_F(\alpha, a, m) = \mu \frac{dZ_2}{d\mu} \quad (2.16a)$$

$$\alpha\beta(\alpha, a, m) = \mu \frac{d\alpha}{d\mu} \quad (2.16b)$$

$$-m\gamma_m(\alpha, a, m) = \mu \frac{dm}{d\mu} \quad (2.16c)$$

$$a\gamma_a(\alpha, a, m) = \mu \frac{da}{d\mu}, \quad (2.16d)$$

which parameterise the dependence on the renormalisation scale μ of the renormalised quantities Z_2 , α , m and a , in terms of α , a and m . The dependence of β , γ , γ_m and γ_a on α , a and m will depend on the renormalisation scheme, but it transpires that in the MS and Weinberg schemes they are independent of the mass, and in the MS scheme, β and γ_m are additionally gauge independent. With the substitutions in eqn (2.16), the above equation becomes

$$\left[\mu \frac{\partial}{\partial \mu} + \alpha\beta \frac{\partial}{\partial \alpha} - m\gamma_m \frac{\partial}{\partial m} + a\gamma_a \frac{\partial}{\partial a} - \frac{n}{2}\gamma_F \right] \Gamma(p_i; \alpha, a, m; \mu) = 0 \quad (2.17)$$

the *renormalisation group (RG) equation*.

The Green's function Γ_0 is not only invariant under a change in μ , as expressed in the RG equation. If we additionally take into account its behaviour under a change in momentum scale—turning up the energy of the beam in an accelerator—then we can use dimensional analysis to get more information about the function.

Denote the mass dimension of Γ by d_Γ . Then we can write

$$\Gamma(p_i; \alpha, a, m; \mu) = \mu^{d_\Gamma} \psi(p_i/\mu; \alpha, a, m/\mu), \quad (2.18)$$

where ψ is a dimensionless function of, crucially, dimensionless arguments. Scaling the momenta by a factor ξ ,

$$\begin{aligned} \Gamma(\xi p_i; \alpha, a, m; \mu) &= \mu^{d_\Gamma} \psi\left(\frac{\xi p_i}{\mu}; \alpha, a, \frac{m}{\mu}\right) \equiv \xi^{d_\Gamma} \left(\frac{\mu}{\xi}\right)^{d_\Gamma} \psi\left(\frac{p_i}{\mu/\xi}; \alpha, a, \frac{m/\xi}{\mu/\xi}\right) \\ &= \xi^{d_\Gamma} \Gamma(p_i; \alpha, a, m/\xi; \mu/\xi), \end{aligned} \quad (2.19)$$

on comparison with eqn (2.18). Differentiating both sides of this by $\xi \partial/\partial \xi$, we find, after a little algebra,

$$\xi \frac{\partial}{\partial \xi} \Gamma(\xi p_i; \alpha, a, m; \mu) = \left(d_\Gamma - m \frac{\partial}{\partial m} - \mu \frac{\partial}{\partial \mu} \right) \Gamma(\xi p_i; \alpha, a, m; \mu).$$

Thus, substituting for $\mu \partial/\partial\mu$ in eqn (2.17),

$$\left[\xi \frac{\partial}{\partial \xi} - d_{\Gamma} - \alpha \beta \frac{\partial}{\partial \alpha} - \gamma_a \frac{\partial}{\partial a} + m(\gamma_m + 1) \frac{\partial}{\partial m} + \frac{n}{2} \gamma_F \right] \times \Gamma(\xi p; \alpha, a, m; \mu) = 0. \quad (2.20)$$

If β and the various γ functions are zero, this reduces to the expression we might have obtained from a naïve scaling argument. Instead, we can see that the presence of interactions, leading to non-zero values for β and γ_i , inevitably leads to violations of scaling symmetry. That is, the process of renormalisation inevitably introduces a *mass scale* of some sort into the theory, whether it be the renormalisation scale μ of dimensional regulation, or the momentum cut-off Λ of Pauli-Villars. Even a massless theory is not immune to this—although the effect of γ_m is suppressed by a zero mass term, the non-zero values of β and γ_i are still present. Because of their rôle in this equation, these functions of the coupling are termed the *anomalous dimensions*.

Equation (2.20) is the fundamental equation of the renormalisation group. In telling us how Γ varies when we change the momentum scale through $p \rightarrow \xi p$, it also tells us how a scaling of p may be compensated by changes in m and α . To make this explicit, we introduce independent functions $f(\xi)$, $\bar{m}(\xi)$, $\bar{a}(\xi)$ and $\bar{\alpha}(\xi)$, via

$$\Gamma(\xi p, \alpha, a, m; \mu) = f(\xi) \Gamma(p, \bar{\alpha}(\xi), \bar{a}(\xi), \bar{m}(\xi); \mu), \quad (2.21)$$

with the boundary conditions $f(1) = 1$, $\bar{m}(1) = m$, $\bar{a}(1) = a$ and $\bar{\alpha}(1) = \alpha$. The terms $\bar{m}(\xi)$, $\bar{a}(\xi)$ and $\bar{\alpha}(\xi)$ are known as the *running mass*, *running gauge parameter* and *running coupling* respectively.

Operating on (2.21) with $\xi \partial/\partial \xi$, we find

$$\xi \frac{\partial}{\partial \xi} \Gamma(\xi p; \alpha, a, m; \mu) = \left[\xi \frac{\partial \bar{\alpha}}{\partial \xi} \frac{\partial}{\partial \bar{\alpha}} + \xi \frac{\partial \bar{a}}{\partial \xi} \frac{\partial}{\partial \bar{a}} + \xi \frac{\partial \bar{m}}{\partial \xi} \frac{\partial}{\partial \bar{m}} + \frac{\xi}{f} \frac{df}{d\xi} \right] \times f(\xi) \Gamma(p; \bar{\alpha}(\xi), \bar{a}(\xi), \bar{m}(\xi); \mu). \quad (2.22)$$

This linear partial differential equation can be solved by standard methods, so that

$$\xi \frac{\partial \bar{\alpha}}{\partial \xi} = \bar{\alpha}(\xi) \beta(\bar{\alpha}) \quad (2.23)$$

$$\xi \frac{\partial \bar{m}}{\partial \xi} = -\bar{m}(\xi) (1 + \gamma_m(\bar{\alpha})) \quad (2.24)$$

$$\xi \frac{\partial \bar{a}}{\partial \xi} = \bar{a}(\xi) \gamma_a(\bar{\alpha}),$$

and

$$\frac{\xi}{f} \frac{df}{d\xi} = d_\Gamma - \frac{n}{2} \gamma_F[\bar{\alpha}(\xi)]. \quad (2.25)$$

with the boundary conditions given above. If all the anomalous dimensions were zero, we would have $f = \xi^{d_\Gamma}$, $\bar{\alpha}$ and \bar{a} independent of ξ , and $\bar{m}(\xi) = m/\xi$, as we might expect from (2.19). Eqn (2.25) can be integrated to give an expression for $f(\xi)$, which can be replaced in (2.21) to give,

$$\Gamma(\xi p; \alpha, a, m; \mu) = \exp \left[d_\Gamma \ln \xi - \frac{n}{2} \int_1^\xi \frac{\gamma_F[\bar{\alpha}(t)]}{t} dt \right] \Gamma(p; \bar{\alpha}(\xi), \bar{a}(\xi), \bar{m}(\xi); \mu). \quad (2.26)$$

This is the solution of the renormalisation group equations, and explicitly tells us how Γ behaves as momenta are scaled. Although this arose from consideration of a change in momentum *scale*, we can use it to predict the behaviour of the Green's functions as we go to *higher external momentum*. Were there no interactions, $\gamma(\bar{\alpha})$ would be zero, and Γ would scale with the naïvely expected scaling factor $\xi^{d_\Gamma} = \exp d_\Gamma \ln \xi$. Again, we see the anomalous dimension $\gamma_F(\bar{\alpha})$ justifying its name by effectively adjusting d_Γ .

We see that the running quantities obey the same differential equations as the plain renormalised parameters of eqn (2.16), so that $\xi \bar{m}(\xi)$ and $m(\mu)$ have the same functional dependence on their arguments.

We now turn to the calculation of the form of $\alpha(\mu)$. From (2.13), we have

$$\alpha = Z_\alpha^{-1} \alpha_0.$$

Now, remembering that $\alpha_0 \propto \mu^{-2\omega}$ and that the only dependence of Z_α on μ is through α , we can write

$$\begin{aligned} \mu \frac{\partial \alpha}{\partial \mu} &= \mu \alpha_0 \frac{-1}{Z_\alpha^2} \frac{\partial Z_\alpha}{\partial \mu} - 2\omega \frac{\alpha_0}{Z_\alpha} \\ &= -\mu \frac{\alpha}{Z_\alpha} \frac{\partial \alpha}{\partial \mu} \frac{\partial Z_\alpha}{\partial \alpha} - 2\omega \alpha. \end{aligned}$$

From eqn (2.16), we can now write

$$\alpha \beta(\alpha) + \alpha^2 \beta(\alpha) \frac{1}{Z_\alpha} \frac{\partial Z_\alpha}{\partial \alpha} + 2\omega \alpha = 0. \quad (2.27)$$

The β -function can be expanded in powers of α , as

$$\beta(\alpha) \equiv \beta_1(\alpha/\pi) + \beta_2(\alpha/\pi)^2 + \dots, \quad (2.28)$$

and since this must be well defined in the limit $\omega \rightarrow 0$, it must be of the form

$$\beta(\alpha) = \beta(\alpha; \omega = 0) + A\omega. \quad (2.29)$$

Replacing this in eqn (2.27), putting $Z_\alpha = (1 + Z_\alpha^{(1)}/\omega + \dots)$ and matching powers of ω , we have

$$0 = A\omega + 2\omega$$

$$0 = \beta(\alpha; \omega = 0) + A\alpha \frac{\partial Z_\alpha^{(1)}}{\partial \alpha},$$

plus a recurrence relation which justifies our assumption that $\beta(\alpha)$ has no poles in ω . Using eqn (2.14) we finally arrive at the result, for QED at $\omega = 0$,

$$\beta(\alpha) \simeq \frac{2}{3} \frac{\alpha}{\pi}$$

We can perform a similar calculation for $m(\mu)$. Using eqn (2.16c), we have

$$\begin{aligned} \gamma_m(\alpha) &= -\frac{\mu}{m(\mu)} \frac{\partial m(\mu)}{\partial \mu} = -\frac{\mu}{m(\mu)} \frac{\partial}{\partial \mu} (m_0 Z_m^{-1}) \\ &= \frac{1}{Z_m} \frac{\partial Z_m}{\partial \alpha} \left[\mu \frac{\partial \alpha}{\partial \mu} = \alpha \beta \right]. \end{aligned}$$

Expanding γ_m as

$$\gamma_m = \gamma_{m,1}(\alpha/\pi) + \gamma_{m,2}(\alpha/\pi)^2 + \gamma_{m,3}(\alpha/\pi)^3 + \dots, \quad (2.30)$$

and Z_m as

$$Z_m = \left[1 + \frac{\alpha(\mu)}{\pi} \frac{Z_{11}}{\omega} + \left(\frac{\alpha(\mu)}{\pi} \right)^2 \left\{ \frac{Z_{22}}{\omega^2} + \frac{Z_{21}}{\omega} \right\} + O(\alpha^3(\mu)) \right]$$

(we shall return to this expansion in eqn (3.27) of chapter 3), we can match powers of ω to obtain

$$\gamma_{m,1} = -2Z_{11}, \quad \gamma_{m,2} = -4Z_{21}, \quad (\beta_1 - \gamma_{m,1})Z_1 = 4Z_{22}. \quad (2.31)$$

The three-loop β and γ_m functions (at $\omega = 0$) are given in table 2 on page 42.

Now that we have the RG β -function, we can use it to find an expression for the running coupling $\alpha(\mu)$. Integrating eqn (2.23), we have

$$-\frac{1}{\alpha} \simeq \frac{\beta_1}{\pi} \ln \xi + (\text{constant}),$$

which, with the boundary condition $\bar{\alpha}(\xi = 1) = \alpha$, becomes

$$\bar{\alpha}(\xi) \simeq \frac{\alpha}{1 - \frac{\beta_1}{\pi} \ln \xi}. \quad (2.32)$$

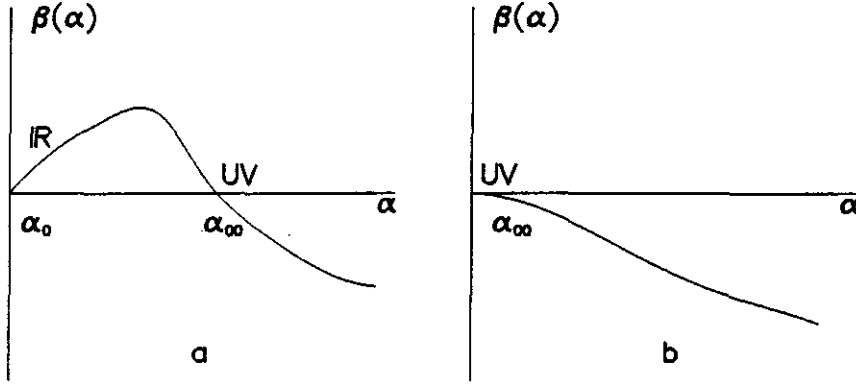


Figure 1 Possible forms for the renormalisation group β -function

Now, $\alpha/\pi < 1$, so that $\bar{\alpha}(\xi)$ starts off at α and increases as ξ increases, indicating that the QED coupling becomes stronger for large momenta or, equivalently, for small distances, until it enters a non-perturbative regime.

For QCD, which we will deal with from now on, the one-loop expression for β is, from table 2,

$$\begin{aligned}\beta_1 &= -\frac{11}{6}C_A + \frac{2}{3}T_F N_F \\ &= -\frac{2}{d} \quad \text{for } \text{SU}(3)_c \\ &= -\frac{9}{2} \quad \text{for } N_f = 3 \text{ flavours,}\end{aligned}$$

where we have defined

$$d = \frac{12}{33 - 2N_F}. \quad (2.33)$$

Because β_1 is negative for $N_F \leq 16$, the strong coupling $\bar{\alpha}_s(\xi)$ becomes weaker at large momenta, and the theory is *asymptotically free*.

The above analysis is possible in general. From eqn (2.23), we can see that if, for a certain value of α , β is negative (say), then α will be driven downwards, and a new value of $\beta(\alpha)$ will be appropriate. By this means, α will approach a stable value as μ goes to the infrared ($\mu = 0$) or ultraviolet ($\mu = \infty$) limits, corresponding to the low and high energy behaviour of the theory. In figure 1, we can see two possible forms for $\beta(\alpha)$: in figure 1a, if α starts in the region where $\beta > 0$, then it will increase to $\alpha(\infty)$ as μ increases, and decrease to $\alpha(0)$ as μ decreases; in figure 1b, α will *decrease* to $\alpha(\infty) = 0$ as μ increases, and go to infinity in the infrared limit. Perturbation theory can provide β only near $\alpha = 0$, but we expect the β -function for QED to be of the form of figure 1a, and that for QCD to be like figure 1b.

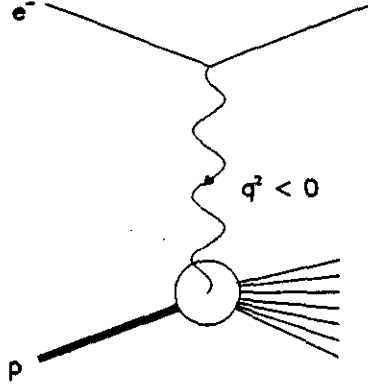


Figure 2 Deep inelastic scattering of photons off protons

A typical use of the RG analysis is to describe the phenomenology of deep inelastic scattering, where a virtual photon of large Euclidean momentum $Q^2 \equiv -q^2 > 0$ is used to probe the structure of the proton (figure 2). In this context, we should renormalise at a scale $\mu \sim \sqrt{Q^2}$, and so render harmless the $\ln Q^2/\mu^2$ terms which appear in calculations. From eqn (2.16b) we obtain

$$\begin{aligned} \frac{1}{2} \ln \mu^2 &= \int^{\alpha_s} \frac{dx}{x\beta(x)} \\ &= \psi(\alpha_s) + \text{constant} \end{aligned} \quad (2.34)$$

which defines $\psi(\alpha)$ up to a constant. We can use this to determine the behaviour of $\alpha_s(\mu)$ at large μ . To one loop, we have $\beta(x) = x\beta_1/\pi$, and we can set the renormalisation group invariant constant of integration to be $\frac{1}{2} \ln \Lambda$. Replacing $\beta_1 = -2/d$, we can easily integrate (2.34) to obtain

$$\alpha_s(\mu) \simeq \frac{\pi d}{\ln(\mu^2/\Lambda^2)}.$$

At two loops, we can again integrate eqn (2.34), This time setting the integration constant to

$$\text{constant} = \frac{1}{2} \ln \Lambda_{\text{MS}}^2 - \frac{\beta_2}{\beta_1^2} \ln \left(-\frac{\beta_1}{2\pi} \right),$$

so that

$$\frac{1}{2} \ln \mu^2 \simeq -\frac{\pi}{\beta_1 \alpha_s} - \frac{\beta_2}{\beta_1^2} \ln \left(-\frac{\beta_1 \alpha_s}{2\pi} \right) + \frac{1}{2} \ln \Lambda_{\text{MS}}^2.$$

Substituting $\beta_1 = -2/d$ [17], we have the two-loop running coupling

$$\alpha_s(\mu) \simeq \frac{\pi d}{\ln(\mu^2/\Lambda_{\text{MS}}^2)} \left[1 + \frac{\beta_2 \ln \ln(\mu^2/\Lambda_{\text{MS}}^2)}{\beta_1^2 \frac{1}{2}(\mu^2/\Lambda_{\text{MS}}^2)} \right]. \quad (2.35)$$

From deep inelastic scattering data, the value of Λ_{MS} is measured to be [18]

$$\Lambda_{\text{MS}} = 240^{+150}_{-120} \text{ MeV}$$

for five flavours. This number should, however, be treated with caution, as the extraction depends strongly on the data used, and on the number of flavours and loops in the analysis, as well as on the method of analysis itself.

For experiments at large Q^2 perturbation theory can be safely used, but as we come down in energy, power corrections appear due to non-perturbative effects in the QCD vacuum, so that perturbation theory breaks down and we enter a phase of the theory in which the quarks are bound into hadrons. Perturbation theory (or, at least, perturbations around free-quark wavefunctions) can tell us nothing about the physics beyond this $Q^2 = \Lambda^2$ boundary. Because it is the checkpoint which marks the border into the hadron jungle, we might expect it to be of the order of the hadron masses, although it is not predicted by the theory. This turns out to be true, with Λ lying between 0.1–0.5 GeV. This scale is present even in the large Q^2 limit when the quark masses can be neglected.

Doing a similar calculation for the mass anomalous dimension γ_m , we end up with

$$m(\mu) \simeq \frac{\hat{m}}{(\frac{1}{2} \ln \mu^2 / \Lambda^2)^{-\gamma_{m,1}/\beta_1}}, \quad (2.36)$$

where we have defined

$$\hat{m} = \lim_{\mu \rightarrow \infty} m(\mu) \left(\frac{-\pi}{\beta_1 \alpha(\mu)} \right)^{-\gamma_{m,1}/\beta_1}, \quad (2.37)$$

This mass \hat{m} is RG invariant (renormalisation point independent), see eg [19]. The invariant mass \hat{m} is scheme dependent in general, as the RS dependences of $m(\mu)$ and $\alpha(\mu)$ will not, in general, cancel one another [19]. However, we can see that the RS dependence can only be multiplicative, so that the *ratio* of invariant masses of quarks of different flavours must be RS independent.

Inserting the two-loop expressions for β and γ_m into eqn (2.24), it is standard to show that [19]

$$m(\mu) \simeq \frac{\hat{m}}{(\frac{1}{2} \ln \mu^2 / \Lambda_{\text{MS}}^2)^{-\gamma_{m,1}/\beta_1}} \left[1 - \gamma_{m,1} \frac{\beta_2 \ln \ln \mu^2 / \Lambda_{\text{MS}}^2}{\beta_1^3 \frac{1}{2} \ln \mu^2 / \Lambda_{\text{MS}}^2} + \frac{1}{\beta_1^2} \left(\gamma_{m,2} - \frac{\gamma_{m,1} \beta_2}{\beta_1} \right) \frac{1}{\frac{1}{2} \ln \mu^2 / \Lambda_{\text{MS}}^2} \right]. \quad (2.38)$$

2.2 Integration by parts

The method of integration by parts is elementary, but the idea behind it is very powerful. We start off with an integral we cannot immediately do and, in the simplest case, replace it by two integrals we can. After each application of the method, we are left with more, simpler, integrals.³ The method is well known: we are simply taking this opportunity to illustrate the method using the notation we use extensively in chapters 3 and 4.

The basic identity, expressed in terms of momenta k and p is

$$\int d^D k \frac{\partial}{\partial k_\mu} [q_\mu \phi(k, p)] = 0$$

where $q \in \{k, p\}$ and ϕ is any scalar function of the momenta k and p .

For example, if we define $I(a, b)$ and $f(a, b)$ through

$$I(a, b) = \int d^D k f(a, b) = \int d^D k \frac{1}{k^{2a}(p+k)^{2b}},$$

then we can find one recurrence relation by putting $q = k$ above, to get

$$\begin{aligned} 0 &= \int d^D k \frac{\partial}{\partial k_\mu} [k_\mu f(a, b)] \\ &= \int d^D k [-b f(a-1, b+1) + b p^2 f(a, b+1) - (2a+b-D) f(a, b)] \\ &= [-b A^- B^+ + b p^2 B^+ - (2a+b-D)] I(a, b) \end{aligned} \tag{2.39}$$

where $A^\pm I(a, b) = I(a \pm 1, b)$, etc. Similarly, with $q = p$, we obtain

$$0 = [p^2(a A^+ - b B^-) + (b A^- B^+ - a A^+ B^-) + (a-b)] I(a, b). \tag{2.40}$$

With the aid of these recurrence relations, and others obtained after operating with $p^2 \partial/\partial p^2$, we can manipulate integrals of the above form. It must be admitted, however, that for such simple integrals this exercise is rather pointless, as we never generate any integrals simpler than the one we started off with. The technique only becomes useful when we consider integrals with more complicated denominator structures, as we will in section 3.3.

2.3 Current algebra

Current algebra is one of the phenomenological theories of the nuclear forces which preceded QCD and the electro-weak theory. Although it has been swallowed by those more formal

³... thus replacing difficulty with tedium. This process can get out of hand.

and complete theories, it is still useful for its rather physical approach to the subject. For a summary, see eg [20].

For convenience, we are using the term ‘current algebra’ rather loosely in this section. In the remainder of this thesis, we will deal almost exclusively with perturbative determinations of the quark mass parameters; spontaneous symmetry breaking (SSB), chiral perturbation theory (ChPT) and the operator product expansion (OPE), on the other hand, are all concerned with non-perturbative sources for the masses, and the language they use is more related to that of current algebra than to perturbation theory. We will review these contributions in the rather distinct sections below.

2.3.1 Current algebra

Current algebra is essentially a phenomenological fit to elementary particle reaction data, with group theory included, and parameterised by a number of constants to be fitted from experiment. It starts with the observation that, in semileptonic electroweak processes, we can split the T-matrix element into a purely hadronic part and a simple electroweak part in the following manner:

$$\begin{aligned}\langle e, b|T|e, a\rangle &= (\bar{u}_e \gamma^\mu u_e) \frac{1}{q^2} \langle b|J_\mu^{\text{em}}(q)|a\rangle, \\ \langle e, b|T|\nu, a\rangle &= (\bar{u}_e \gamma^\mu (1 - \gamma_5) u_\nu) \langle b|J_\mu^{\text{wk}}(q)|a\rangle\end{aligned}\tag{2.41}$$

where a and b represent hadronic states, q the 4-momentum transfer to the hadrons, and J^{wk} and J^{em} , defined by these equations, represent the weak and electromagnetic *currents*, analogous to currents in the classical limit. See figure 3. In the first of the equations (2.41),

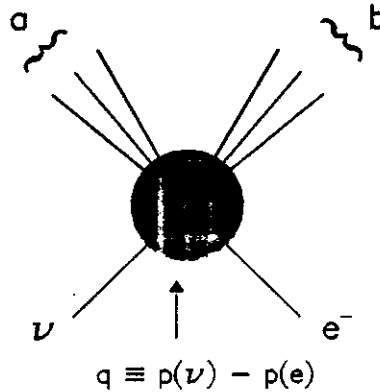


Figure 3 A weak interaction $\langle b|J_\mu^{\text{wk}}|a\rangle$.

notice that we have a $1/q^2$ term which we recognise as a propagator—there is no such term for the weak interaction, which was taken to happen at a point. The terms γ_μ and $\gamma_\mu \gamma_5$

are present in the weak expression because only vector and axial interactions could describe Fermi and Gamow-Teller interactions (respectively) with the correct helicities for the interacting particles; parity considerations then fix the term to be a $V - A$ interaction, with $\gamma^\mu(1 - \gamma_5)$.

The currents J^{wk} and J^{em} can be decomposed into

$$\begin{aligned} J_\mu^{\text{wk}} &= \frac{G}{\sqrt{2}}(V_\mu - A_\mu) \\ J_\mu^{\text{em}} &= \alpha(V_3^\mu + V_8^\mu/\sqrt{3}) \end{aligned} \quad (2.42)$$

where V_μ and A_μ are Lorentz iso- and axial-vectors respectively. That they are also objects in the space of $\text{SU}(3)$ generators was one of the insights of current algebra, and essentially says that the currents are an *approximate* representation of $\text{SU}(3)$, or that V_3 , say, carries properties corresponding to the third member of an $\text{SU}(3)$ octet, or the third member of an isospin triplet.

The time components V_i^0 and A_i^0 of the currents obey the algebra

$$\begin{aligned} [V_i^0, V_j^0] &= if_{ijk}V_k^0 \\ [V_i^0, A_j^0] &= if_{ijk}A_k^0 \\ [A_i^0, A_j^0] &= if_{ijk}V_k^0 \end{aligned} \quad (2.43)$$

where f_{ijk} are the structure constants of $\text{SU}(3)$. The two currents together therefore generate the direct product group, (chiral) $\text{SU}(3) \otimes \text{SU}(3)$, described in more detail in section 2.3.4 below. The hadronic currents J can be taken to be composed of lepton-like ‘bare’ quark currents [21]

$$\begin{aligned} j_\mu^i &= \bar{q} \frac{1}{2} \lambda^i \gamma_\mu q, \\ j_{\mu 5}^i &= \bar{q} \frac{1}{2} \lambda^i \gamma_\mu \gamma_5 q, \end{aligned} \quad i = 1, \dots, 8$$

which describe quarks, and which are the currents we shall refer to below. Currents with other Lorentz structures can be defined, and are useful in PCAC, below.

Indirectly from the algebra, one can obtain *sum-rules*, integral relations which must be obeyed by physical states. By fitting these to experiment, one can extract quark *current masses* of the order of [22]

$$m_u \approx m_d \approx 7 \text{ MeV}, \quad m_s \approx 156 \text{ MeV}.$$

One may also use $\text{SU}(6)$ symmetry to relate the matrix elements to observables [23] and obtain

$$\tilde{m} \equiv \frac{1}{2}(m_u + m_d) = 5.4 \text{ MeV}$$

for the mean up and down mass. Finally, through PCAC relations like

$$-(m_u + m_d)\langle\bar{\psi}_u\psi_u + \bar{\psi}_d\psi_d\rangle \sim f_\pi^2 m_\pi^2,$$

one can obtain current mass ratios [22]

$$\frac{m_d}{m_u} \approx \frac{7}{4}, \quad 23 \lesssim \frac{m_s}{\tilde{m}} \lesssim 28.$$

Although there is some variation in the values these different methods give for the quark masses, they all give values well below the *constituent masses* we would naïvely expect by halving meson masses (say).

Also note that, since $m_u \approx m_d$, we can say that SU(2) is approximately conserved, and the difference $m_d - m_u$ is a measure of the extent to which SU(2) is broken. One might object that the difference may be small, but the ratio of the masses $m_d : m_u \sim 2$ is rather large, and that it is surely this ratio which should be the true measure of symmetry breaking. However, quarks typically have energies of the order of the strong interaction scale, much greater than quark current masses, so that the ratio of the mass to the total energy is almost the same for both quarks, making the difference between, rather than the ratio of, the quark masses the better measure. Similarly, the more substantial difference $m_s - \tilde{m}$ is a measure of the extent to which SU(3) is broken, but when the *s* quark energy is much larger than $m_s \sim 150 \text{ MeV}$, we can expect no large violation of SU(3) symmetry, and no substantial flavour asymmetries. The ‘strong interaction scale’ is not a particularly well-defined quantity; there are a number of quantities, such as Λ , f_π , M_ρ , M_ρ , which have the dimensions of mass and which are finite in the chiral limit. Which of these we choose when we want a numerical value for the scale is, to some extent, a matter for personal preference, but Λ and f_π are rather too small—if we need a value, we shall use $M_\rho = 770 \text{ MeV}$.

As a final point, we will remark that the masses of current algebra are not supposed to be inertial masses of free quarks, but instead chiral-symmetry breaking parameters with the dimensions of mass. One may, in fact, find the value of the parameters directly from symmetry breaking effects and obtain $15 \text{ MeV} \lesssim \tilde{m} \lesssim 40 \text{ MeV}$, consistent with the above values to the extent that they are much smaller than constituent masses. The subject of chiral symmetry is taken up below.

2.3.2 Spontaneous symmetry breaking

In the sections which follow, we will use the idea of a broken vacuum symmetry. Spontaneous symmetry breaking (SSB), arises out of the Goldstone theorem, which we shall now briefly

describe. We shall avoid the technical details of the theorem, and of spontaneous symmetry breaking, since they are not themselves relevant here, and we shall simply describe the mechanism rather informally, and refer the reader to any textbook for the details.

Given an eigenstate $|n\rangle$, such that $H|n\rangle = E|n\rangle$, and a charge Q , which is conserved $i\dot{Q} = [Q, H] = 0$, we can see that

$$HQ|n\rangle = EQ|n\rangle$$

so that the conserved charge Q generates a multiplet of eigenstates of equal energy to $|n\rangle$. This is a *manifest* symmetry, also known as the Wigner-Weyl realisation of a symmetry, and has the physical consequence of producing a particle spectrum broken into multiplets of equal mass. The fact that the observed particles are classifiable into multiplets of particles with approximately equal mass suggests that they are representations of an approximate manifest symmetry.

If $|n\rangle = a_n^\dagger|0\rangle$ is a one-particle H -eigenstate, then

$$Q|n\rangle = [Q, a_n^\dagger]|0\rangle + a_n^\dagger Q|0\rangle$$

will also be a one-particle eigenstate if $Q|0\rangle = 0$, since $[Q, a_n^\dagger]$ has the same form as a_n^\dagger , by virtue of the algebra. In this situation, the state $|0\rangle$ is the unique vacuum. If $Q|0\rangle \neq 0$, on the other hand, the state $Q|0\rangle$ will be more complicated and, specifically, the states

$$|\theta\rangle = \exp(-iQ\theta)|0\rangle$$

will *all* be zero-energy eigenstates, like $|0\rangle$, and the operator Q can be seen to be associated with the generation of zero-mass particles. This is confirmed by the more formal arguments of the Goldstone theorem.

To develop a field theory which has a charge which does not annihilate the vacuum, and which therefore generates a degenerate continuum of vacua, we (or nature) must arbitrarily choose one of the vacuum states, $|\phi_0\rangle$, take it to be the physical vacuum, and expand the physical states around this one. In doing so, we have either 'broken' or 'hidden' the symmetry, in the sense that the physical states are not representations of the fundamental symmetry group of the system. The symmetry is still present, however, and manifests itself in zero-mass 'excitations' of the physical vacuum into one of the other original degenerate vacuum states.

The original Lagrangian might be expressed in terms of the field ϕ (with any Lorentz and group indices suppressed), which has a non-vanishing vacuum expectation value (VEV) and a degenerate vacuum. We choose one of the vacua ϕ_0 , which is invariant under a

subgroup of the invariance group of the original Lagrangian, and express the field ϕ as $\phi = \phi_0 + \chi$. Now $\langle \phi \rangle = \langle \phi_0 \rangle \neq 0$, $\langle \chi \rangle = 0$, and χ , rather than ϕ , is the physical field. When we re-express the Lagrangian in terms of χ , we find that this shift has changed the mass terms in the Lagrangian, leaving χ with some massive and some massless degrees of freedom. The massless degrees of freedom, the Nambu-Goldstone (NG) bosons, are the remnant of the symmetry of the original Lagrangian.

What we have described here is static SSB, through the Goldstone mechanism. This can be contrasted with dynamical SSB, developed by Nambu and Jona-Lasinio, in which the process of dressing the quark breaks a chirally symmetric Lagrangian, and generates a $q = 0$ pole in the axial vertex which corresponds to the massless pseudoscalar NG boson.

2.3.3 PCAC

Our next topic in this review of Current Algebra is PCAC, Partial Conservation of the Axial Current. This was a phenomenologically motivated assumption with numerous applications. We may start from pion decay

$$\langle 0 | j_{\mu 5}^i(x) | \pi^j(q) \rangle = \delta^{ij} f_\pi q_\mu e^{-iqx}, \quad (i, j = 1, 2, 3)$$

where the pion decay constant $f_\pi = 93.3 \pm 0.1 \text{ MeV}$ can be extracted from $\pi^+ \rightarrow \mu^+ \nu$, and j^i are the SU(2) axial currents. Taking the divergence of this, and going on the pion mass shell $q^2 = m_\pi^2$, we have

$$\langle 0 | \partial^\mu j_{\mu 5}^i(0) | \pi^j \rangle = -i \delta^{ij} f_\pi m_\pi^2. \quad (2.44)$$

The pion mass m_π is small but non-zero, so that the axial current is not quite conserved. The vector current *is* conserved, so that the charges generating flavour SU(3)_f are constants, and we see SU(3)_f as a manifest symmetry. Conversely, the non-conservation of the axial current is linked to SSB, and a non-zero quark VEV. Part of the point of the description of PCAC in this thesis is to do with how this last quantity depends on the mass m_π .

The statement that $\partial j_5 \approx 0$ in the operator sense, is the Nambu statement of PCAC. We may derive from this the alternative version of PCAC in which, for the transition $a \rightarrow b$,

$$\langle b | \partial j_5(q) | a \rangle \approx f_\pi \frac{(\text{amp. for } a \rightarrow b + \pi)}{q^2 - m_\pi^2},$$

so that the transition is described by a pion pole dominating a smoothly varying background.

Pion PCAC gives rise to the Goldberger-Treiman relationship between the pion decay constant and the axial couplings. This latter relation is experimentally in error by about 6%.

Kaon PCAC, in which we use eqn (2.44) with $i = 4, 5$, is a more approximate symmetry, due to the larger mass m_K , and the corresponding Goldberger-Treiman relation is in error by 10–30% [24].

In discussing the model-dependent assumptions behind the VEV's $\langle \bar{q}q \rangle$, Scadron [24] has distinguished strong from neutral PCAC, as follows. *Strong PCAC*, which is the standard version, assumes that the quarks transform under SU(3) in the simple way described in Gell-Mann's early paper [21], in which $\langle i | \bar{q} \lambda^j q | k \rangle \propto d^{ijk}$, and assumes that the vacuum is SU(3) symmetric, so that $\langle \bar{u}u \rangle = \langle \bar{d}d \rangle = \langle \bar{s}s \rangle$. These assumptions lead to the current-quark mass ratio

$$\frac{m_s}{\tilde{m}} = 2 \frac{m_K^2}{m_\pi^2} - 1 \approx 25 \quad (\text{Strong PCAC}), \quad (2.45)$$

which uses the information that the meson masses are proportional to the squares of the quark masses, and leads to the mass values

$$\tilde{m} \approx 5 \text{ MeV}, \quad m_s \approx 150 \text{ MeV} \quad (\text{Strong PCAC}).$$

Strong PCAC also implicitly assumes that the quark VEV does not vanish in the chiral limit, so that

$$\langle \bar{q}q \rangle = O(1) \quad (\text{Strong PCAC}).$$

Scadron criticises this scheme because it takes no account of the spectroscopic successes of the non-relativistic SU(6) model, and he proposes an alternative.

In *Neutral PCAC*, Scadron makes a distinction between the light current quark fields, and the fully dressed constituent-quark fields, and claims that this is significant for chiral symmetry breaking. By describing hadrons in terms of essentially free current quarks, he obtains a mass formula in which the meson masses are of the order of a single power of quark masses, and develops an intricate argument to show that

$$\frac{m_s}{\tilde{m}} = \sqrt{2 \frac{m_K^2}{m_\pi^2} - 1} \approx 5, \quad (\text{Neutral PCAC})$$

so that

$$\tilde{m} \approx 56\text{--}62 \text{ MeV}, \quad m_s \approx 310 \text{ MeV}. \quad (\text{Neutral PCAC})$$

This alternative formalism also demands that

$$\begin{aligned} \langle \bar{q}q \rangle &= O(m_q) \\ \langle \bar{u}u \rangle / m_u &= \langle \bar{d}d \rangle / m_d = \langle \bar{s}s \rangle / m_s. \end{aligned} \quad (\text{Neutral PCAC}) \quad (2.46)$$

and has the result that pion pole dominance is substantially weaker than in the conventional picture, so that one might expect substantial deviations from conventional current algebra.

There is some experimental support for this picture, as it accounts for deviations from the Goldberger-Treiman relation, and for the anomalously large πN σ -term, rather better than conventional PCAC. Improved measurements and an improved understanding of chiral symmetry breaking have tended to make these deviations smaller [25], but still the main objection to neutral PCAC is eqn (2.46), which seems to suggest that the condensates break chiral symmetry whilst being themselves zero in the chiral limit. There is no contradiction here, however, as the neutral scheme is associated with a non-vanishing connected 4-quark condensate [25]

$$\int_x T(\bar{q}q(0)\bar{q}q(x))_{conn} = O(1), \quad (q = u, d, s)$$

which leads to the Goldstone theorem. Although this is consistent, many people remain uneasy that the 2-quark condensate vanishes when there is no particular reason for it to do so, and so this scheme is mentioned in reviews (eg [22]) without ever having become part of PCAC doctrine.

There are several other schemes, similar to this, in which $\int_x \langle (\bar{q}q)^p \rangle = O(1)$ ($p \geq 1$), and the lower order terms vanish identically. Although these schemes cannot be ruled out on any fundamental grounds, they rapidly become implausible.

2.3.4 Chiral perturbation theory

Although we will not directly use chiral perturbation theory (ChPT) in this thesis, we will briefly review it here, to provide a context for the discussion of quark masses later on. In this section we shall see that the phenomenological current masses m_q , which we may identify with the theoretical running mass $m(\mu)$, should not be regarded as an inertial mass, but rather as a chiral-symmetry breaking parameter with the dimensions of mass.

The QCD Lagrangian of appendix D.1 has a good deal of symmetry as it stands. If we set quark masses to zero, and so study the limit $m_u = m_d = m_s = 0$, the symmetry group of the Lagrangian grows substantially. It would grow even more if we set the masses of the heavy charm, bottom and top quarks to zero, but these quarks are far too heavy for this to be a useful approximation, so we go to the other extreme and give them infinite masses so that their degrees of freedom freeze out, and they can be removed from the effective theory. It is slightly surprising that this chiral theory, with only one dimensionful parameter Λ , is still a reasonable approximation to reality.

With the light quark masses zero (and avoiding the technicalities of gauge fixing and ghost corrections), the Lagrangian becomes

$$\mathcal{L} = \sum_{q=u,d,s} \bar{q} i \not{D} q - \frac{1}{4} \text{Tr} F_{\mu\nu} F^{\mu\nu}.$$

This Lagrangian has a global $U(3) \otimes U(3)$ symmetry, in that it is invariant under the flavour transformations

$$\begin{aligned} q_i &\rightarrow [\exp i\alpha_a \lambda_a]_{ij} q_j \\ q_i &\rightarrow [\exp i\beta_a \lambda_a \gamma_5]_{ij} q_j \end{aligned} \quad a = 0, 1, \dots, 8. \quad (2.47)$$

The direct product $U(3) \otimes U(3)$ factors into $SU(3) \otimes SU(3) \otimes U(1) \otimes U(1)$, with the dynamics principally in the $SU(3) \otimes SU(3)$ subgroup. The extra $U(1)$ vector symmetry corresponds to the transformation $q_i \rightarrow e^{i\alpha_0} q_i$, which corresponds in turn to baryon number conservation. The extra $U(1)$ *axial* symmetry, corresponding to $q_i \rightarrow e^{i\beta_0 \gamma_5} q_i$, has no such interpretation: the symmetry cannot be simply realised either manifestly or spontaneously without unphysical predictions. In fact, the axial $U(1)$ symmetry *is* spontaneously broken, but not in the simple way in which the $SU(L)$ symmetry is broken. This problem, the $U(1)$ problem, has been present since the earliest days of QED and QCD, and centres round the anomalous divergence $\partial j_5^0 \neq 0$, and the question of whether or not the associated η_0 pseudoscalar is a Goldstone boson. That the divergence does not vanish suggests that the η_0 is not a Goldstone boson, but one can construct a conserved Q_5^0 such that $Q_5^0|0\rangle \neq 0$, suggesting that it *is*. This confused situation seems to have been resolved only fairly recently, when Witten suggested [26] that the extra $U(1)$ boson could have a (mass)² of order $1/N_c$, in an expansion in terms of the number of colours. This means that in the large N_c chiral limit of $m_q \rightarrow 0$ and $N_c \rightarrow \infty$, the anomaly disappears, the extra boson corresponding to the $U(1)_A$ symmetry is a genuine NG boson, and we are left with L^2 genuine Nambu-Goldstone bosons of $U(L) \otimes U(L)$ breaking. Despite this, Scadron [24] claims that because the QCD vacuum is so complicated, it is clearer to approach the problem through the (dynamical) Nambu mechanism of SSB, which allows him to state that the extra boson is unambiguously *not* a NG boson. For reviews, see [24,25]. We will confine ourselves to broken chiral $SU(3) \otimes SU(3)$ below.

In the chiral limit of this latter theory, we can say that, due to eqn (2.47), the currents

$$\begin{aligned} V_\mu^a &= \bar{q} \gamma_\mu \frac{1}{2} \lambda^a q \\ A_\mu^a &= \bar{q} \gamma_\mu \gamma_5 \frac{1}{2} \lambda^a q \end{aligned} \quad a = 1, \dots, 8$$

are conserved, so that there are both vector and axial symmetries to be realised in nature.

If a chiral symmetric theory is to be realistic, then the ground state cannot itself be symmetric. If it were, we would expect to see equal-mass particle pairs of opposite parity.

The vacuum $|0\rangle$, then, is not symmetric under $SU(3) \otimes SU(3)$. The symmetry may be broken explicitly by the introduction of non-zero quark masses, to give us chiral perturbation theory (ChPT), or broken spontaneously, to give us Goldstone bosons.

The vector symmetry is not broken. If it were, we would see the Goldstone bosons as a multiplet of light scalars in the meson spectrum, which we do not. Instead we see hadrons in (nearly) mass degenerate multiplets, indicating that the vector symmetry is realised manifestly.

The axial symmetry is spontaneously broken. The eight lightest mesons are pseudoscalars, so that π , K and η are identified as the Goldstone bosons of the broken $SU(3)_A$. The members of this 0^- meson octet do not have zero mass, as Goldstone bosons should, because chiral symmetry is only approximate. The notion of an approximate symmetry was first made precise by Gell-Mann [21] by the assumption that the algebra of the charges is still valid at equal time.

In chiral perturbation theory, we explicitly break the symmetry of the otherwise chiral-invariant QCD Hamiltonian by adding a mass term

$$\begin{aligned}\mathcal{H}_1 &= m_u \bar{u}u + m_d \bar{d}d + m_s \bar{s}s \\ &= \frac{1}{3}(m_u + m_d + m_s)(\bar{u}u + \bar{d}d + \bar{s}s) + \frac{1}{2}(m_u - m_d)(\bar{u}u - \bar{d}d) \\ &\quad + \frac{1}{3}(m_s - \tilde{m})(2\bar{s}s - \bar{u}u - \bar{d}d),\end{aligned}\tag{2.48}$$

and expanding about the $m_q = 0$ limit. Written in the form of eqn (2.48), we can see that the first term is an $SU(3)$ scalar; the second breaks isospin symmetry and is suppressed by the small amount $\frac{1}{2}(m_u - m_d) \sim 1 \text{ MeV}$; and the third term, which transforms under $SU(3)$ like λ_8 , breaks $SU(3)$. The third term is suppressed by $\frac{1}{3}(m_s - \tilde{m}) \sim 50 \text{ MeV}$, which is substantially smaller than the interaction scale, Λ . In this form, it is natural to interpret the ‘masses’ m_q as chiral symmetry-breaking parameters.

2.3.5 The operator product expansion

The QCD Lagrangian has an essentially simple form which, through perturbation theory, leads to simple ultraviolet behaviour which matches well with results from deep inelastic scattering (for example), thus tending to support both QCD itself and the validity of the method of perturbation theory. The fact of asymptotic freedom, giving rise to the simplicity of the high energy theory, and the apparent fact of confinement, giving rise to a force strong enough to hold quarks together and capable of producing such a rich hadron spectrum from the result, mean that the low energy theory must be much more complicated than the high energy one. Shifman, Vainshtein and Zakharov (SVZ) were amongst the first to suggest that the infrared theory did not arise because of the breakdown of the perturbation series

in α_s , but because of the emergence of non-vanishing vacuum expectation values (VEV's, or condensates) of quark and gluon operators suppressed by inverse powers of momenta.

Previous attempts to gain access to infrared QCD had relied on phenomenological assumptions which their proponents hoped would be justified by a complete theory— attempts since include direct simulation of QCD on the lattice. SVZ instead suggested [27] starting with the simple high energy theory and using the VEV's to probe the resonances at low energy. Although some of the parameters are fixed from experiment, the theory itself springs from first principles, using QCD to relate the physical resonances to the VEV's.

We start from the Operator Product Expansion (OPE [28]) of the T-product of two currents j , labelled by some index Γ

$$\begin{aligned} T_{\Gamma}(q)\Pi^{\Gamma}(q^2) &= i \int dx e^{iqx} \langle T j_{\Gamma}(x) j_{\Gamma}^{\dagger}(0) \rangle \\ &= \sum_n C_n \langle O_n \rangle. \end{aligned} \quad (2.49)$$

Here, $\Pi(q^2)$ is a scalar vacuum polarisation, and T_{Γ} is a polynomial in q with the Lorentz structure demanded by the currents on the right hand side. SVZ studied the vector current $j_{\nu} = \bar{q}_i \gamma_{\mu} q_j$ with $J^{PC} = 1^{--}$, but in general we can insert any of the currents which couple to the observed meson states. The O_n are the *local* operators which produce the condensates when they are sandwiched between vacuum states. Since they have mass dimension $d \geq 0$, the coefficients C_n have mass dimension ≤ 0 and the OPE can be regarded as an expansion in powers of (large) external momentum Q^2 . Only those operators with zero Lorentz spin contribute to VEV's, and higher dimension operators $d > 6$ can be neglected, as they are suppressed by ever larger powers of momentum. We can give a complete set of such operators as [27]

$$\begin{aligned} O_0 &= \mathbb{I} & (d=0) \\ O_M &= \bar{\psi} M \psi & (d=4) \\ O_G &= G_{\mu\nu}^a G_a^{\mu\nu} & (d=4) \\ O_{\sigma} &= \bar{\psi} \sigma_{\mu\nu} t^a \widetilde{M} \psi G_a^{\mu\nu} & (d=6) \\ O_{\Gamma} &= \bar{\psi} \Gamma_1 \psi \bar{\psi} \Gamma_2 \psi & (d=6) \\ O_f &= f^{abc} G_{a\mu}^{\nu} G_{b\nu}^{\lambda} G_{c\lambda}^{\mu} & (d=6) \end{aligned}$$

where ψ and G are the quark and gluon fields, M and \widetilde{M} are mass matrices in flavour space, t^a are the colour SU(3) matrices, and the Γ_i are objects in colour, flavour and Lorentz space. The VEV's of these operators must be found from experiments, but they should be universal.

The calculation of the right-hand side of eqn (2.49) can be informally described as being done by an extension of the normal perturbative technique, so that the term $C_G \langle O_G \rangle$,

for example, consists of all those vacuum polarisation Feynman diagrams which include two gluons appearing from the vacuum. For example, the expression for the heavy-fermion self-energy in the OPE is

$$\begin{aligned}
 -i\Sigma_{\text{OPE}} = & \left[\text{---}\!\!\!\!\!\rightarrow + \text{---}\!\!\!\!\!\overset{\text{wavy}}{\curvearrowright} + \dots \right] \mathbb{I} \\
 & + \left[\text{---}\!\!\!\!\!\overset{\text{wavy}}{\curvearrowright} + \dots \right] \langle \bar{\psi} M \psi \rangle \\
 & + \left[\text{---}\!\!\!\!\!\overset{\text{gluon}}{\curvearrowright} + \dots \right] \langle G_{\mu\nu}^a G_{\mu\nu}^a \rangle
 \end{aligned}$$

where only the $O(\alpha_s)$ terms have been shown. From this it follows (i) that the coefficient C_0 multiplying the unit operator is simply the usual expression obtained in the high energy theory, and the only one to survive when the higher-dimension condensates are suppressed, and (ii) that the OPE can be generally interpreted as confining its non-perturbative features to the operators O_n , leaving the coefficients C_n to be calculated perturbatively. There are thus *two* expansions implicit within the OPE.

Although the OPE is valid to all orders in perturbation theory (where it was introduced [28] as a technical device), it breaks down in resonance physics as the condensates become infrared stable. This happens at $O(Q^{-11})$, and above this critical dimension, it must be abandoned [27]. At this point, SVZ used instanton solutions, specific to QCD.

When the VEV's are put into the OPE, we have QCD's prediction for the vacuum polarisation operator. This can be obtained independently by a dispersion relation from observable cross sections. This equality is a sum rule, and permits a fairly direct experimental test of QCD.

Applications of the non-perturbative OPE have been successfully made to systems of equal mass quarks, and of light quarks. Applications to heavy-light systems suffer from large corrections in the series (2.49) [29].

2.4 Effective field theory of the infinite mass quark

In section 4.4 below, we will relate work we have done on quark wavefunction renormalisation to a recent attempt to develop an effective field theory for infinite-mass 'static' quarks (EFT). Before we do this, it is appropriate to review this topic here. The review will be rather swift, but will pave the way for new results in section 4.4.

The Effective Field Theory (EFT) of the infinite mass quark has a pedigree which springs from both phenomenology and lattice calculations. It is related to the quenched

approximation of lattice gauge theory, but recent work (building on earlier work on an effective field theory for non-relativistic QED) has changed it from an approximation into an analytic theory. In this relatively recent area, reviews are scarce, but Eichten and Hill [30] give a clear account of the field theory, and Bjorken [31] gives a phenomenologically motivated account. The infinite mass limit has also been strongly promoted [31] as a model-independent starting point for the calculation of physical amplitudes, at which a number of predicted weak matrix elements simplify, in particular model-independent calculations [32] of Cabbibo-Kobayashi-Maskawa (CKM) matrix elements. The latter are a particularly good example, as the last CKM angles to be reliably calculated are those for transitions between the heavy quarks for which the theory is a good approximation.

For several processes involving these heavy quarks, for example those involved in the calculation of f_B , the large rest mass of the heavy quark is not deposited in kinetic energy of lighter hadrons. This observation leads to the approximation that the heavy quark is a static colour source. In fact, ‘motionless’, or ‘constant-momentum’, would be better terms than ‘static’, as momentum is conserved at vertices involving the quarks and their colour is not fixed, just like ordinary dynamical quarks; the term is conventional, however, so that we shall continue to use it, with this reservation.

There are several ways of moving toward the formal limit of QCD in which the heavy quark masses are taken to be infinite: the authors of ref [30] simply wrote down the static EFT Lagrangian in Minkowski space as

$$\mathcal{L}_M = b^\dagger (i\partial_0 + gA_0) b,$$

where b is the two component field of the heavy quark. The heavy antiquark field is quite independent in this formalism—a four component field which describes both retains trivial dependence on the heavy mass. This leads to the free Minkowski space propagator

$$\frac{i}{p_0 + i\epsilon},$$

and a gauge field which participates only through its zeroth component, the interaction of which with the quark is simply $-g$ times a gauge group generator.

The fields have only trivial components in spin space. That is, the quark’s spin is *decoupled* from its dynamics⁴, so that we find extra SU(2) spin symmetries generated by the quarks and antiquarks, and a consequent mass degeneracy in hyperfine multiplets. As well as this, the strong dynamics ignores the flavour labels of the heavy quarks $c, b, t \dots$, so that there is a flavour symmetry as well, giving a (flavour) \otimes (spin) symmetry comparable [31] to nuclear physics’ Wigner symmetry.

⁴... or, more physically, the hyperfine coupling of a heavy-light system in QCD falls to zero as the heavy quark mass increases to infinity

This EFT is well defined; it is also a good approximation to the physical world if $m_Q \gg \Lambda_{QCD}$. We can make an expansion in $1/M$ about the infinite mass limit by expanding in $(p^\mu - (m_Q, 0))/M_Q$, with model-dependent coefficients. This corresponds to a heavy quark nearly at rest, and nearly on shell [30].

There are some doubts about the renormalisability of the EFT, but the main limitation to it is in calculations in which a small mass or energy difference is significant [30,32]. Because the EFT neglects heavy mass differences, it can give incorrect results if used carelessly in situations such as this.

2.5 The mass mess

In the process of renormalising QED to one loop, above, we introduced several mathematical quantities which play the rôle of masses. There are a number of other mass parameters which may be introduced.

The most obvious quark and electron mass is the *constituent mass*, M^{cons} . In the case of QED, this is the experimental mass of the electrified particles of the cathode rays, which has been known (or at least its ratio with the elementary charge) for some time [33]. In the case of QCD, it is the naïve mass obtained by assuming that the baryons are very simply composed of quarks, and then dividing the mass of a proton, say, by three.

Following on from the remarks at the end of section 2.1.4, we can define a ‘pole mass’, M , by demanding that the bare Feynman propagator

$$S_F(p) \equiv \frac{i}{\not{p} - m_0 - \Sigma(p)}$$

has a pole as $p^2 \rightarrow M^2$. The term $\Sigma(p)$ is the proper self-energy, which we write in the form

$$\Sigma(\not{p}) = m_0 A(p^2/m_0^2) + (\not{p} - m_0) B(p^2/m_0^2).$$

Combining these two, we obtain

$$iS_F^{-1} = (1 - B) \left[\not{p} - m_0 \left(1 + \frac{A(p^2)}{1 - B(p^2)} \right) \right],$$

and we define a scale-dependent and gauge-dependent *effective mass* [34,35]

$$m^{\text{eff}}(p^2) = m_0 \left(1 + \frac{A(p^2)}{1 - B(p^2)} \right), \quad (2.50)$$

so that the inverse propagator becomes

$$iS_F^{-1} = (1 - B(p^2))(\not{p} - m^{\text{eff}}(p^2)). \quad (2.51)$$

This has a pole in \not{p} , and can be made to have residue i by adjustment of $B(p^2)$.

In QED, the pole is at $p^2 = M^2$, such that

$$Z_M^{-1} \equiv \frac{M}{m_0} = 1 + \frac{A(p^2 = M^2)}{1 - B(p^2 = M^2)} \quad (2.52)$$

and we can use eqn (2.9) to get the one-loop answer:

$$Z_M^{-1} \equiv \frac{M}{m_0} = 1 + \frac{g_0^2}{(4\pi)^{D/2}} C_F \frac{D-1}{D-3} \Gamma(\omega),$$

where $C_F = 1$ for QED, and $C_F = (N_c^2 - 1)/2N_c$ for a gauge group $SU(N_c)$.

The electron or quark wavefunction can be renormalised at the same time, to ensure that the propagator has a residue of i at this pole, and the renormalised propagator S_r is just $i/(\not{p} - M)$. This scheme is perfectly well-defined in QED, but runs into subtle and deep problems in QCD, which are the subject of the discussion below, and in chapters 3 and 4.

Repeating the argument in terms of renormalised quantities, we may start with the renormalised propagator

$$iS_r^{-1} = \not{p} - m(\mu) - \Sigma(p^2, \mu)$$

and, if we express $\Sigma(p^2, \mu)$ in the form

$$\Sigma(p^2, \mu) = m(\mu)A(p^2, \mu) + (\not{p} - m(\mu))B(p^2, \mu),$$

we can extract the *same* effective mass as

$$m^{\text{eff}}(p^2) = m(\mu) \left(1 + \frac{A(p^2, \mu)}{1 - B(p^2, \mu)} \right), \quad (2.53)$$

so that the inverse propagator becomes

$$iS_r^{-1} = (1 - B(p^2))(\not{p} - m^{\text{eff}}(p^2)).$$

That the effective masses in eqns (2.50) and (2.53) are in fact the same quantity can be seen by observing that

$$1 - B(p^2, \mu) = Z_2(1 - B_0)$$

$$1 - B(p^2, \mu) + A(p^2, \mu) = Z_2 Z_m(1 - B_0 + A_0).$$

The pole mass, M , defined above for QED, is defined [19] in QCD as the value of the momentum which gives a pole in S_r :

$$m^{\text{eff}}(M^2) = M, \quad (2.54)$$

and is thus RS and RP independent. These effective masses have the property that they approach fixed ratios at high Q^2 , and that [35]

$$\lim_{Q^2 \rightarrow \infty} \frac{m_i^{\text{eff}}(Q^2)}{m_j^{\text{eff}}(Q^2)} = \frac{m_{0,i}}{m_{0,j}}.$$

The effective mass $m^{\text{eff}}(Q^2)$ is a candidate for the theoretical quantity which corresponds to the constituent quark mass M^{cons} . It can be interpreted [35] as the struck parton mass in the OPE analysis of lepton-hadron scattering; or we may connect the effective mass of a quark q with the constituent mass by considering the smeared e^+e^- cross section, which will have thresholds in the energy when there is sufficient available to create the lightest $\bar{q}q$ meson. Through this, we can define a mass m^{cons} (also called a ‘constituent mass’) through [34]

$$m_q^{\text{cons}} = m^{\text{eff}}(Q^2 = 4(m_q^{\text{cons}})^2).$$

This parameter matches the physical constituent mass M^{cons} only for heavy quarks, for which $\alpha_s(Q = 2m^{\text{eff}}(Q^2))$ is small. For light quarks, $m_{u,d}^{\text{cons}}$ is very α_s -dependent, but can be estimated [34] to lie in the range 350–400 GeV.

Tarrach [19] has criticised this definition, as it leads to a definition of m^{cons} which is gauge-dependent. A similar definition, in terms of the running mass, $m^{\text{cons}} = m(Q = 2m^{\text{cons}})$, is scheme dependent. Instead, he proposed the simple identification of the constituent mass and pole mass:

$$M^{\text{cons}} = M.$$

This is the identification we make in this thesis. Chapter 3 is devoted to the calculation of the relation between the pole mass and the running mass $m(M)$ renormalised at the scale of the pole mass.

There is also a non-perturbative contribution to the quark mass. Politzer [35] obtained the expression

$$(\text{full mass}) = m(M) \left(\frac{\alpha_s(Q)}{\alpha_s(M)} \right)^d + \langle \bar{\psi}\psi \rangle \frac{16\pi\alpha_s(Q)}{Q^2} \left(\frac{\alpha_s(Q)}{\alpha_s(M)} \right)^{-d}, \quad (2.55)$$

in the Landau gauge (it is gauge dependent) and for three flavours, and used it to conclude that the masses of the heavy quarks (c and b) were principally perturbative, that the masses of the light quarks (u and d) were principally non-perturbative, and that the mass of the

strange had substantial contributions from both perturbative and non-perturbative sources. We return to this subject in chapter 3.

The usual interpretation of the running mass $m(\mu)$ is that it corresponds to the current-algebra mass discussed in section 2.3. It is less usual to identify the pole mass M and the constituent mass M^{cons} , but this identification is the one we make in this thesis.

This interpretation is substantially more problematic in QCD than it is in QED. The lighter quark masses produced by current algebra are of the order of [22]

$$m_u \approx m_d \approx 7 \text{ MeV}, \quad m_s \approx 156 \text{ MeV},$$

whilst the constituent masses we would predict, on the assumption that the masses of the hadrons are entirely due to the masses of the quarks inside them, would be

$$M_u^{\text{cons}} \approx M_d^{\text{cons}} \approx \frac{M_{\text{proton}}}{3} \approx 310 \text{ MeV} \quad M_s^{\text{cons}} \approx \frac{M_{\phi}}{2} \approx 480 \text{ MeV}.$$

Roughly the same masses are obtained [19] from e^+e^- thresholds, or from magnetic moments.

An explanation of this immediate failure is that there can be no ‘physical’ mass of a quark if the quark cannot be isolated and weighed, so the quark is never on-shell, so the notion of an on-shell mass, which is what the pole mass is, becomes rather abstract. This does not stop us defining such a mass, and remarking that for heavy quarks (which have a mass well above the scale μ) we can make a non-relativistic approximation, and say that the quarks are *nearly* on-shell. This would lead us to suppose that the disparity between the constituent and current masses, or between the running and pole masses, should become smaller as we come to examine heavier quarks.

This turns out to be true, as the running masses for the heavy quarks, obtained from e^+e^- data, are [22]

$$m_c(M_c^{\text{cons}}) = 1.27 \pm 0.05 \text{ GeV}, \quad m_b(M_b^{\text{cons}}) = 4.25 \pm 0.10 \text{ GeV}.$$

and the corresponding constituent masses are

$$M_c^{\text{cons}} \approx \frac{M_{\eta_c}}{2} \approx 1.5 \text{ GeV} \quad M_b^{\text{cons}} \approx \frac{M_{\Upsilon}}{2} \approx 4.7 \text{ GeV},$$

showing much better agreement.

Despite all this, the pole mass is not entirely useless for light quarks, as it is the mass parameter which is used in bag models [36]. For heavy quarks, too, it is the pole mass which is to be used in the Balmer formula.

The mass parameters may be mutually related. This will be done in chapter 3, where the calculation of the ratios relating M , m_0 , and $m(M)$ is described.

The various mass parameters are summarised in table 1.

<i>mass</i>		Gauge indep?	Scale indep?	Scheme indep?
M	pole mass is the position of the pole in the fermion propagator, defined by $S_f^{-1}(p^2 = M^2) = 0$. After renormalisation, we have $m_0 = Z_M M$. [section 2.1.4]	✓	✓	✓
\hat{m}	The RG invariant mass , which appears as a constant of integration when integrating the gamma function γ_m . [section 2.1.5]	✓	✓	\times^1
$m(\mu)$	The MS renormalised mass is the fermion mass which has been made finite by the application of the renormalisation prescription. It is defined through $m_0 = Z_m m$, where Z_m is a Laurent series in the regulation parameter $\omega \equiv (4 - D)/2$. [section 2.1.2]	✓	\times	\times
$m^{\text{eff}}(p^2)$	effective mass defined in QCD by eqn (2.50) so that S_F has a denominator $\not{p} - m^{\text{eff}}(p^2)$. [section 2.5]	\times	\times	\times
m_0	Bare mass is the mass parameter which appears in the Lagrangian of appendix D.1. It is divergent.	✓	✓	✓

Table 1 The principal quark mass parameters in renormalised field theories, ordered in increasing acceptability as physical parameters. (1): \hat{m} is RS dependent, but this dependence must be multiplicative (see eqn (2.37)), so that one can conclude [19] that the ratio of invariant masses for different flavours must be RSI.

2.6 The wavefunction mess

Fermion wavefunction renormalisation is not bedevilled with the same problems of interpretation as mass renormalisation. This is partly due to the fact that the wavefunction (or, equivalently, the propagator) is not directly observable, but some proportion of the problem with mass renormalisation is due to the mass scale which inevitably creeps in with regulation.

We have alluded to wavefunction renormalisation above, giving the definition of the renormalised propagator, S_r in (2.11). In this section, which is a prelude to the calculations of chapter 4, we will briefly describe how the renormalisation constant Z_2 is calculated and make mention of the gauge dependence of renormalisation constants.

We start from the bare Feynman propagator, and demand that it has a pole at $\not{p} = M$, with residue Z_2 ,

$$S_f = \frac{Z_2}{\not{p} - M} + (\text{finite at } \not{p} = M),$$

or, comparing with eqn (2.51),

$$\frac{i}{S_f} \equiv \not{p} - m_0 - \Sigma(\not{p}) = (1 - B) (\not{p} - m^{\text{eff}}(p^2)).$$

The pole is at $\not{p} = M \equiv m^{\text{eff}}(p^2 = M^2)$. Expanding in \not{p} about M , we find

$$\not{p} - m^{\text{eff}}(p^2) = (\not{p} - M) \left(1 - \frac{\partial m^{\text{eff}}}{\partial \not{p}} \Big|_{\not{p}=M} \right) = \frac{i}{S_f(1 - B)}.$$

Thus, directly,

$$iZ_2^{-1} = (1 - B) \left(1 - \frac{\partial m^{\text{eff}}}{\partial \not{p}} \right) \Big|_{\not{p}=M} \quad (2.56)$$

where $\partial m^{\text{eff}}/\partial \not{p} \Big|_{\not{p}=M} = 2 \partial \ln m^{\text{eff}}/\partial \ln p^2 \Big|_{p^2=M^2}$, and

$$\frac{\partial \ln m^{\text{eff}}}{\partial \ln p^2} = \frac{A' - B'}{1 + A - B} + \frac{B'}{1 - B}$$

with $A' = \partial A/\partial \ln p^2$. Expanding A and B as before, we have $A = \sum_n \Omega^n A_n(p^2)$, with $\Omega \propto (p^2)^{-\omega}$ (and similarly for B). This gives us

$$\begin{aligned} A'(p^2) &= p^2 \frac{\partial A}{\partial p^2} \\ &= -\omega \Omega A_1 + \Omega A'_1 - 2\omega \Omega^2 A_2 + \Omega^2 A'_2 + O(\omega^2, \Omega^3). \end{aligned}$$

The techniques by which we can calculate the coefficients A_i and B_i to the two-loop order are described in chapter 3, and we return to this calculation, using those techniques, in chapter 4.

The Johnson-Zumino identity [37] guarantees that the dimensionally regulated photon wavefunction renormalisation constant is gauge invariant. The same argument fails in QCD, and there is no general proof that Z_2 is gauge invariant in that theory.

Quantity	General gauge theory	$N_C = 3$	$d = 12/(33 - 2N_F)$	ref
β_1	$-\frac{11}{6}C_A + \frac{2}{3}T_F N_F$	$-\frac{11}{2} + \frac{1}{3}N_F$	$-2/d$	[16]
β_2	$\frac{1}{2} \left[-\frac{17}{6}C_A^2 + \frac{5}{3}C_A T_F N_F + C_F T_F N_F \right]$	$-\frac{1}{4}(51 - \frac{19}{3}N_F)$	$\frac{107}{8} - \frac{19}{2}/d$	[41,42]
β_3	$\frac{1}{32} \left[-\frac{2857}{54}C_A^3 + \frac{1415}{27}C_A^2 T_F N_F \right.$ $\left. - \frac{158}{27}C_A T_F^2 N_F^2 + \frac{205}{9}C_A C_F T_F N_F \right.$ $\left. - \frac{44}{9}C_F T_F^2 N_F^2 - 2C_F^2 T_F N_F \right]$	$-\frac{1}{32} \left[\frac{2857}{2} - \frac{5033}{18}N_F + \frac{325}{54}N_F^2 \right]$	$\frac{37117}{768} - \frac{243}{16}/d - \frac{325}{48}/d^2$	[2]
$\gamma_{m,1}$	$\frac{3}{2}C_F$	2	2	[16]
$\gamma_{m,2}$	$\frac{3}{16}C_F^2 + \frac{97}{48}C_A C_F - \frac{5}{12}C_F T_F N_F$	$\frac{101}{12} - \frac{5}{18}N_F$	$\frac{23}{6} + \frac{5}{3}/d$	[19]
$\gamma_{m,3}$	Unknown	$\frac{1}{32}(1249 - \frac{2216}{27}N_F$ $+ \frac{160}{3}N_F \zeta(3) - \frac{140}{81}N_F^2)$	$\frac{55}{2}\zeta(3) - \frac{2591}{144}$ $- (10\zeta(3) - \frac{313}{12})/d - \frac{35}{18}/d^2$	[3]

Table 2 Coefficients of the expansion of the renormalisation group functions β and γ , defined in eqns 2.28 and 2.30 respectively. The expressions in the third column are for $SU(3)_c$, and were obtained by setting $C_A = N_C = 3$, $C_F = (N_C^2 - 1)/2N_C = 4/3$ and $T_F = \frac{1}{2}$. QED corresponds to the group $U(1)$, and can be obtained from these results by the substitutions $C_A = 0$, and $C_F = T_F = N_F = 1$. Those in the fourth column are in terms of d of eqn (2.33).

Chapter 3

3-loop Relation of Quark $\overline{\text{MS}}$ and Pole Masses

In the previous chapter, we have seen how to regulate and renormalise a theory, and how we use the renormalisation group to define the running mass, $m(\mu)$. We have also defined the pole mass, M .

In this chapter, we use these ideas to calculate the relation between these two masses to the next-to-leading order. To *leading* order, the relation is

$$m(\mu) \approx M \left[\frac{\alpha_s(\mu)}{\alpha_s(M)} \right]^d \quad (3.1)$$

where the renormalised strong coupling to lowest order is, from section 2.1.5,

$$\alpha_s(\mu) \simeq \frac{\pi d}{\ln(\mu^2/\Lambda^2)} \quad (3.2)$$

and $d = 12/(33 - 2N_F)$, as given in eqn (2.33). The leading order correction to m/M was found in [38], but the next-to-leading corrections have never been calculated before for massive propagators. This $O(\alpha_s^2)$ calculation is not made difficult by the combinatorial explosion which complicates higher order calculations—there are only six two-loop diagrams to calculate; nor by any intrinsic complication of a non-Abelian theory—the extra diagrams are relatively simple to calculate. This calculation is difficult because of the analytic complication of the integrals which appear when we deal with massive fermions, rather than massless ones. The integrals involved are horrendous (for a foretaste, see eqn (3.15)), and we deal with them by using a combination of integration by parts, analytical ingenuity¹, and large amounts of CPU time.

We use integration by parts (cf section 2.2)—a technique which was first used in this area by [1], but which we have extended. The method was first applied to this particular problem by Grafe in [8], with some errors. We have applied computer algebra to the problem,

¹...my supervisor's, who is blessed with a horrifying talent for integrals, and the enviable ability to extract delight from evaluating them.

using REDUCE [39], and we find that the complicated integrals which appear in this two-loop calculation can eventually be reduced, by recurrence relations which we derive, to a large number of simple integrals plus one spectacularly difficult one, which was analytically obtained by Broadhurst in [40].

The material of this chapter was first published [43] by D J Broadhurst and myself of the Open University, and K S Schilcher and W Grafe of the University of Mainz.

3.1 Sources of radiative correction

To relate M and $m(\mu)$ in eqn (3.1), we need the three ratios

$$\frac{m(\mu)}{m(M)}, \quad \frac{m(M)}{M}, \quad \frac{\alpha_s(\mu)}{\alpha_s(M)}. \quad (3.3)$$

To find $\alpha_s(M)$, given its value at some other scale μ , we integrate the RG relation (2.16) to get

$$\ln \frac{\mu^2}{M^2} = \int_{\alpha_s(\mu)}^{\alpha_s(M)} \frac{dx}{\tilde{b}(x)}$$

where

$$\begin{aligned} \tilde{b}(x) &\equiv \frac{x\beta(x)}{-2} = \frac{x}{-2} \left(\beta_1 \frac{x}{\pi} + \beta_2 \left(\frac{x}{\pi} \right)^2 + \beta_3 \left(\frac{x}{\pi} \right)^3 + \dots \right) \\ &= \frac{x^2}{\pi d} - \frac{\beta_2}{2} \frac{x^3}{\pi^2} - \frac{\beta_3}{2} \frac{x^4}{\pi^3} + \dots \end{aligned}$$

with $\beta_1 = -2/d$. Now changing variables $x \mapsto t = x/\pi d$, the above expression becomes

$$\ln \frac{\mu^2}{M^2} = \int_{\alpha_s(\mu)/\pi d}^{\alpha_s(M)/\pi d} \frac{dt}{b(t)} \quad (3.4)$$

where

$$\begin{aligned} b(t) &= t^2 + \sum_{n=1}^{\infty} b_n t^{n+2}, \\ b_1 &= -d^2 \beta_2 / 2 = -\frac{107}{16} d^2 + \frac{19}{4} d, \\ b_2 &= -d^3 \beta_3 / 2 = -\frac{37117}{1536} d^3 + \frac{243}{32} d^2 + \frac{325}{96} d. \end{aligned}$$

We now want to do the same for m —that is, to find the running mass after a change of scale from μ to M , or find $m(M)$, given $m(\mu)$. We can obtain this relation from the definition of $\gamma_m(\alpha)$, in eqn (2.16). Defining for convenience $\tilde{\alpha} = \alpha_s/(\pi d)$,

$$\frac{\partial m}{\partial \tilde{\alpha}} = \frac{\partial m}{\partial \mu} / \frac{\partial \tilde{\alpha}}{\partial \mu} = -\frac{m \gamma_m}{\tilde{\alpha} \beta}$$

so that

$$\ln \frac{m(M)}{m(\mu)} = - \int_{\tilde{\alpha}(\mu)}^{\tilde{\alpha}(M)} d\tilde{\alpha} \frac{\gamma_m}{\tilde{\alpha}\beta}. \quad (3.5)$$

The γ_m and β function are expanded as

$$\gamma_m = \gamma_{m,1}\tilde{\alpha}d + \gamma_{m,2}(\tilde{\alpha}d)^2 + \gamma_{m,3}(\tilde{\alpha}d)^3$$

$$\beta = \beta_1\tilde{\alpha}d + \beta_2(\tilde{\alpha}d)^2 + \beta_3(\tilde{\alpha}d)^3$$

so that, using the binomial expansion to expand the denominator

$$\tilde{\alpha}\beta = \tilde{\alpha}^2\beta_1d(1 + \beta_2/\beta_1(\tilde{\alpha}d) + \beta_3/\beta_1(\tilde{\alpha}d)^2),$$

we obtain the series

$$-\frac{\gamma_m}{\tilde{\alpha}\beta} = \xi_1/\tilde{\alpha} + \xi_2 + \xi_3\tilde{\alpha}$$

$$\xi_1 = -\gamma_{m,1}/\beta_1 = d$$

$$\xi_2 = -d(\gamma_{m,2}\beta_1 - \gamma_{m,1}\beta_2)/\beta_1^2$$

$$\xi_3 = -d^2(\gamma_{m,3}\beta_1^2 - \gamma_{m,2}\beta_2\beta_1 + \gamma_{m,1}(\beta_2^2 - \beta_1\beta_3))/\beta_1^3.$$

Thus, integrating (3.5) and exponentiating the result,

$$\frac{m(M)}{m(\mu)} = \frac{c(\alpha_s(M)/\pi d)}{c(\alpha_s(\mu)/\pi d)} \quad (3.6)$$

where

$$\begin{aligned} c(\tilde{\alpha}) &= \exp[\xi_1 \ln \tilde{\alpha} + \xi_2 \tilde{\alpha} + \frac{1}{2}\xi_3 \tilde{\alpha}^2] \\ &= \tilde{\alpha}^{\xi_1} (1 + \xi_2 \tilde{\alpha} + \frac{1}{2}(\xi_2^2 + \xi_3)\tilde{\alpha}^2). \end{aligned}$$

Putting in the expressions for ξ_i above, and then the expressions for the renormalisation constants, given in table 2 on page 42, we obtain the series

$$c(\tilde{\alpha}) = \tilde{\alpha}^d + \sum_{n=0}^{\infty} c_n \tilde{\alpha}^{d+n}$$

where

$$\begin{aligned} c_1 &= \frac{23}{12}d^2 + (\frac{5}{6} - b_1)d, \\ &= \frac{107}{16}d^3 - \frac{17}{6}d^2 + \frac{5}{6}d \end{aligned}$$

and

$$\begin{aligned} c_2 &= (\frac{55}{8}\zeta(3) - \frac{2591}{576})d^3 - (\frac{5}{2}\zeta(3) - \frac{313}{48})d^2 - \frac{1}{2}(\frac{35}{36} + b_2)d + \frac{1}{2}c_1(c_1 - b_1), \\ &= \frac{11449}{512}d^6 + \frac{5243}{1536}d^5 - \frac{33985}{9216}d^4 + (\frac{55}{8}\zeta(3) - \frac{73}{64})d^3 \\ &\quad - (\frac{5}{2}\zeta(3) - \frac{1841}{576})d^2 - \frac{35}{72}d. \end{aligned}$$

For the third ratio in (3.3), we need the relation between the pole mass M and the running mass there, $m(M)$,

$$\frac{M}{m(M)} = 1 + \sum_{n=1}^{\infty} d_n \left(\frac{\alpha_s(M)}{\pi d} \right)^n \quad (3.7)$$

with

$$\begin{aligned} d_1 &= \frac{4}{3}d \\ d_2 &= Kd^2 \end{aligned} \quad (3.8)$$

in which the leading term d_1 was given by [22,38], and the next-to-leading term d_2 is calculated in this chapter.

The coefficient d_2 is much harder to calculate than any of $\{b_1, b_2, c_1, c_2, d_1\}$. Those terms in the set which cannot be obtained by fairly elementary methods can be found by the highly-developed techniques of integration by parts for massless diagrams—techniques which have been used successfully for massless three [2,3,4], four [5] and five-loop [6,7] counterterms². The corresponding techniques for massive diagrams are rudimentary by comparison, and were apparently first used by Grafe [8] in 1983. In this chapter we extend his methods and correct some errors in his results.

The reason the coefficient is so difficult to calculate is that, in diagrams 4b and d, there are three intermediate heavy quarks, and thus *three* terms which involve the external momentum p , in the denominator of the integrations over the internal momenta k_i . These terms are of the general form $((p + k_i)^2 - m^2)^n$.

To find d_2 , we must evaluate the integrals corresponding to the diagrams of fig 4. Only the one-loop diagram can reasonably be done by hand,³ the two-loop diagrams are too analytically difficult. Using integration by parts, outlined in sections 2.2 and 3.3, we can reduce these analytically complicated integrals to many simpler ones. This needs computer algebra if it is to be done reliably—we used REDUCE [39].

3.2 Reduction to on-shell integrals

To find d_2 , we first obtain an expression for the pole mass M in terms of m_0 , a_0 and g_0 : the bare mass, gauge parameter and coupling constant of the unrenormalised theory. The pole mass is defined by the condition that the unrenormalised Feynman propagator

²The results in [5] have recently been shown [44] to be in error, with an incorrect coefficient for the $O(\omega^0, \alpha^3)$ term. Errors in [6,7] have also been reported [45]. This does not affect our results.

³Grafe did them *all* by hand, which probably accounts for the error.

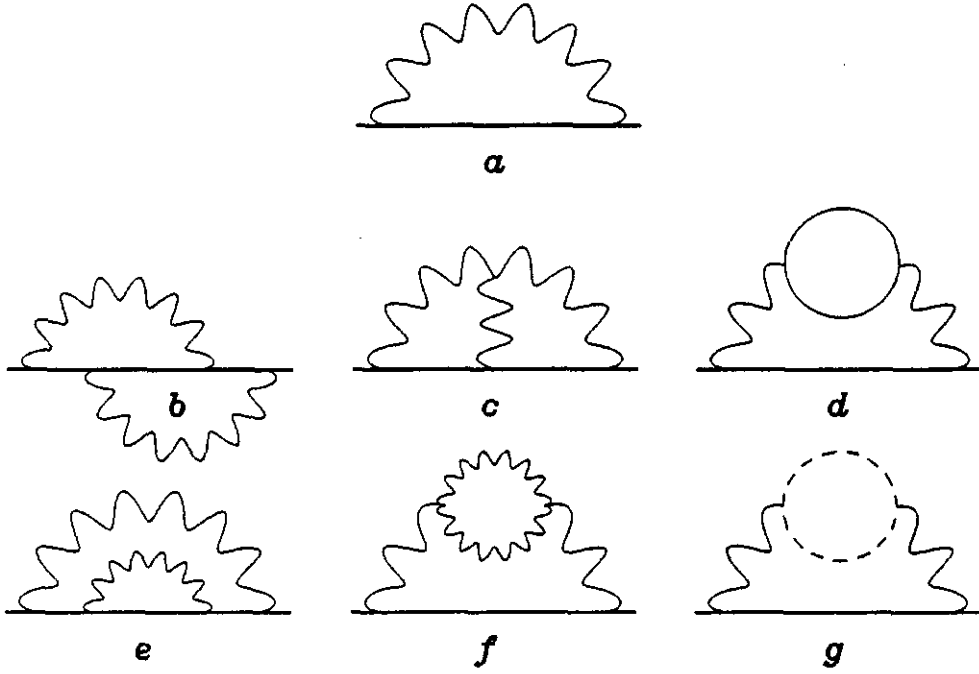


Figure 4 The one and two loop quark self energy diagrams

$$S_F(p) \equiv \frac{i}{\not{p} - m_0 - \Sigma(p)}$$

has a pole as $p^2 \rightarrow M^2$. The term $\Sigma(p)$ is the proper self energy, obtained, to two loops, by summing the diagrams of fig 4. We choose to expand it as follows:

$$\Sigma(p) = \sum_{n=1}^{\infty} \left[\frac{g_0^2}{(4\pi)^{D/2} p^{2\omega}} \right]^n \left[m_0 A_n(m_0^2/p^2) + (\not{p} - m_0) B_n(m_0^2/p^2) \right].$$

By plugging this expression into the denominator of $S_F(p)$, expanding about $m_0^2/p^2 = 1$ in a Taylor series, and setting that denominator to zero, we obtain the expansion

$$m_0 = Z_M M = M \left(1 + \sum_{n=1}^{\infty} \left[\frac{g_0^2}{(4\pi)^{D/2} M^{2\omega}} \right]^n C_n \right) \quad (3.9)$$

with

$$C_1 = -A_1(1), \quad (3.10a)$$

$$C_2 = -A_2(1) + A_1(1) \left[A_1(1) + 2A_1'(1) - B_1(1) \right]. \quad (3.10b)$$

The terms $A_1(1)$, $A_1'(1)$ and $B_1(1)$ are obtained from the one-loop diagram 4a. We shall work through the one-loop calculation in rather tedious detail—the two-loop calculation is the same in principle, only longer. Much longer. Using the Feynman rules of Appendix D.2,

$$-i \Sigma(p) \Big|_{1\text{loop}} = \text{diagram showing a horizontal line with momentum } p \text{ entering from the left and } p+k \text{ exiting to the right, with a wavy loop labeled } k \text{ attached to the line.}$$

$$\begin{aligned}
&= g_0^2 C_F \int \frac{d^D k}{(2\pi)^D} \gamma_\mu \frac{1}{\not{p} + \not{k} - m_0} \gamma_\nu \frac{1}{k^2} \left[g^{\mu\nu} + (a_0 - 1) \frac{k^\mu k^\nu}{k^2} \right], \quad (3.11) \\
&= -i(m_0 A(p) + (\not{p} - m_0) B(p))
\end{aligned}$$

so that

$$\text{Tr } \Sigma = 4m_0(A - B), \quad (\text{Tr } \not{p} \Sigma) \Big|_{p^2=m_0^2} = 4m_0^2 B. \quad (3.12)$$

Now we use the trace theorems of Appendix D.3 to obtain the on shell results

$$\begin{aligned}
\text{Tr } \Sigma &= -4ig_0^2 m_0 C_F \frac{1}{\mu^{2\omega}} (D + a_0 - 1) I(1, 1; m_0) \\
\text{Tr } \not{p} \Sigma \Big|_{p^2=m_0^2} &= -2ig_0^2 C_F \frac{1}{\mu^{2\omega}} [(D - 2) I(0, 1; m_0) - 2m_0^2 (D + a_0 - 3) I(1, 1; m_0)]
\end{aligned}$$

(this is eqn (2.8) with $p^2 = m_0^2$ and $I(\alpha, \beta; p) = 0$ for $\beta \in \mathbb{Z} \leq 0$). Using the on shell limit for $I(\alpha, \beta; p)$ given in eqn (E.8), and the expressions for A and B given above in (3.12), we have

$$\begin{aligned}
A_1(1) &= C_F \frac{D-1}{D-3} \Gamma(\omega) \\
B_1(1) &= -C_F \frac{a_0}{D-3} \Gamma(\omega)
\end{aligned} \quad (3.13)$$

We will also need the term $A'_1(1)$. To find it, we define the quantity

$$\begin{aligned}
\overline{\Sigma} &\equiv \frac{1}{4} \text{Tr } i(1 + \Delta \not{p})(-i\Sigma) \\
&= \Omega[(A_1 - B_1)m_0 + \Delta p^2 B_1]
\end{aligned}$$

where Δ is an arbitrary parameter which we will use to extract parts of the differentiated expression, and

$$\Omega \equiv g_0^2 \Gamma(\omega) / (4\pi)^{D/2} p^{2\omega}$$

which will be used, eventually, as an expansion parameter. Noting that $\partial\Omega/\partial p^2 = -\omega/p^2 \times \Omega$, we have

$$\frac{\partial \overline{\Sigma}}{\partial p^2} \Big|_{p^2=m_0^2} = \Omega \Big|_{p^2=m_0^2} [-\omega(A_1 - B_1)/m_0 - \omega \Delta B_1 - (A'_1 - B'_1)/m_0 + \Delta(B_1 - B'_1)]$$

(where $A_1 = A_1(1)$, etc). We differentiate the integrand of eqn (3.11) (carefully) and go on shell to obtain an expression for $\partial \overline{\Sigma}/\partial p^2$ in terms of $I(\alpha, \beta; m_0)$'s. Setting $\Delta = 1/m_0$ in the above expression gives us A'_1 in terms of A_1 , B_1 and I 's, which we can invert to get

$$A'_1(1) = \frac{1}{2} C_F \left(D - 1 - \frac{a_0}{D-3} \right) \Gamma(\omega). \quad (3.14)$$

Setting $\Delta = 0$ gives B'_1 , but we don't need this at this stage. Comparing eqns 3.10, 3.13 and 3.14, we can see that the gauge dependence of $B_1(1)$ and $A'_1(1)$ cancels in C_2 .

The calculation of the gauge invariant term $A_2(1)$ involves the six two-loop diagrams 4b–4g, and requires the techniques of the next section. When we come to calculate these two-loop diagrams, we come across integrands with denominators similar to, but more complicated than, those in eqn (3.11).

These integrals over loop momenta are such that the numerators of the integrands may be expressed as polynomials in the same Lorentz scalars as appear in the denominators, allowing cancellations and consequent simplification of the integrals. Thus we are left with a large number of primitive scalar integrals, which we evaluate *on the bare mass shell*, at $m_0^2/p^2 = 1$.

3.3 Integration by parts

We now show how to extend to massive integrals the method of integration by parts of [1].

All of the two-loop integrals generated by the procedure of the previous section are of the form

$$\iint_{\mathbb{M}^D} \frac{d^D k_1 d^D k_2}{k_1^{2\alpha_1} k_2^{2\alpha_2} (k_1^2 + 2p \cdot k_1)^{\alpha_3} (k_2^2 + 2p \cdot k_2)^{\alpha_4} [(k_1 + k_2)^2 + 2p \cdot (k_1 + k_2)]^{\alpha_5}} \equiv \pi^D (p^2)^{D-\Sigma\alpha_i} N(\alpha_1, \dots, \alpha_5), \quad (3.15)$$

or

$$\iint_{\mathbb{M}^D} \frac{d^D k_1 d^D k_2}{k_1^{2\alpha_1} (k_1 - k_2)^{2\alpha_2} k_2^{2\alpha_3} (k_1^2 + 2p \cdot k_1)^{\alpha_4} (k_2^2 + 2p \cdot k_2)^{\alpha_5}} \equiv \pi^D (p^2)^{D-\Sigma\alpha_i} M(\alpha_1, \dots, \alpha_5). \quad (3.16)$$

In order to evaluate these integrals, we use recurrence relations to reduce them to sums of simpler integrals and a single irreducibly hard one.

The method we use is that of integration by parts, which was briefly described in section 2.2. The key identity is

$$\iint d^D k_1 d^D k_2 \frac{\partial}{\partial k^\mu} [q^\mu f(k_1, k_2, p)] = 0 \quad (3.17)$$

where $k \in \{k_1, k_2\}$, $q \in \{k_1, k_2, p\}$ and f is any scalar function of the Minkowski loop momenta $k_{1,2}$ and the external momentum p . This identity generates six recurrence relations for a general two-loop integral.

If we let $k = p$, as well, we obtain three more, only two of which are independent. We cannot derive this latter case from eqn (3.17) by itself. To derive it, we consider the function

$$f(p, k, l) = \prod_{i=1}^5 a_i^{-\alpha_i}$$

where the a_i are the *Minkowski* invariants in the denominator of eqn (3.15) and we may define $\Sigma \equiv \sum^5 \alpha_i$. By dimensional analysis, we have

$$\int \int_{\mathbb{M}^D} dk dl q_\mu f(p, k, l) = p_\mu (p^2)^{D-\Sigma} K$$

where $q \in \{p, k, l\}$ and K is a dimensionless number, independent of p^2 . Thus

$$\begin{aligned} \frac{\partial}{\partial p_\mu} \int \int dk dl q_\mu f(p, k, l) [p^2]^{\Sigma-3D/2} \\ = \frac{\partial}{\partial p_\mu} (p^2)^{-D/2} K \\ = DK(p^2)^{-D/2} + p_\mu K[-2p^\mu (p^2)^{-D/2-1} \cdot D/2] \\ = 0. \end{aligned}$$

This means that we can consistently say

$$0 = \int \int dk dl \frac{\partial}{\partial r_\mu} [q_\mu f(p, k, l) [p^2]^{\Sigma-2D/2}], \quad r, q \in \{p, k, l\} \quad (3.18)$$

All nine possibilities are shown in tables 6 and 7 on page 62. Not all of the eight independent relations for each of (3.15) and (3.16) are particularly useful. For example, in both tables, the relations with $q = p$ are unhelpful in that they raise one index without lowering any other in return (be reminded that we want to use these relations to lower selected indices in N and M as far as possible, so that they may be reduced to simpler integrals). With these considerations in mind, we discover that the most useful of the relations are

$$\left(2\alpha_2 + \alpha_4 + \alpha_5 - D + \alpha_4 4^+ 2^- + \alpha_5 5^+ [2^- - 3^-] \right) \times N(\alpha_1, \dots, \alpha_5) = 0 \quad (3.19)$$

$$\begin{aligned} \left(2\alpha_2 + \alpha_1 + \alpha_4 - D + \alpha_1 1^+ [2^- - 3^-] + \alpha_4 4^+ [2^- - 5^-] \right) \\ \times M(\alpha_1, \dots, \alpha_5) = 0 \quad (3.20) \end{aligned}$$

where $1^\pm N(\alpha_1, \dots, \alpha_5) \equiv N(\alpha_1 \pm 1, \alpha_2, \dots, \alpha_5)$, etc. The first is from eqn (3.17) with $k = q = k_2$ and the second is a linear combination of two of the relations of table 7, with $k = k_1$ and $q = k_1 + k_2$.

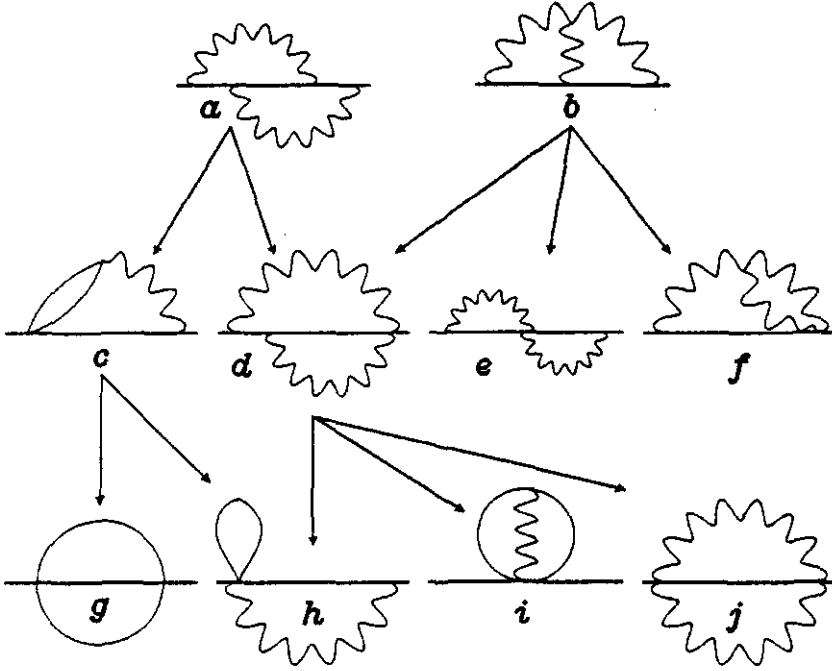


Figure 5 Illustration of the use of the recurrence relations, eqns (3.19) and (3.20)

The REDUCE program which implements these recurrence relations is reproduced in appendix C.

In fig 5 we illustrate the application of these relations to diagrams 5a and 5b, which represent the general structures of equations (3.15) and (3.16). In this figure, the gluon-like lines correspond to gluon-like denominators of the general form k_i^{2n} in the integrands, and the quark-like lines to terms like $(k_i^2 + 2p \cdot k)^n$. We represent the disappearance of the terms from the denominator by the disappearance of the corresponding line from the diagram. The figure is generated by applying eqn (3.19) to diagrams 5a and 5c, and applying eqn (3.20) to diagrams 5b and 5d. Diagrams 5e, 5f, 5h and 5j are easily evaluated as products of one-loop diagrams. For example, diagram 5e is

$$\text{Diagram 5e} = \left(\text{Diagram 5d} \right)^2$$

which is related to the integral $(I(\alpha, \beta; m_0)/\mu^{2\omega})^2$, (compare eqns (E.1) and (E.8)).

The bubble diagram 5i is not so easy. This diagram is related to the integral $M(0, \alpha, 0, \beta, \gamma)$. For arbitrary p ,

$$M(0, \alpha, 0, \beta, \gamma) = \frac{1}{\pi^D (p^2)^{D-\alpha-\beta-\gamma}} \times \left[\iint_{\mathbb{M}^D} \frac{dk_1 dk_2}{(k_1 - k_2)^{2\alpha} [(p + k_1)^2 - p^2]^\beta [(p + k_2)^2 - p^2]^\gamma} \right].$$

Now, substitute $k = k_1 - k_2$ and $l = p + k_2$, to give

$$M = \frac{1}{\pi^D (p^2)^{D-\alpha-\beta-\gamma}} \int_{\mathbb{M}^D} \frac{dk}{k^{2\alpha}} \int_{\mathbb{M}^D} \frac{dl}{[(k+l)^2 - p^2]^\beta (l^2 - p^2)^\gamma}.$$

Following the working of eqn (E.9) and including a Feynman parameter x , we find

$$M = \frac{i(-)^\omega}{\pi^{D/2} (p^2)^{D-\alpha-\beta-\gamma}} \frac{\Gamma(\beta + \gamma - D/2)}{\Gamma(\beta)\Gamma(\gamma)} \times \int_0^1 dx (1-x)^{D/2-\beta-1} x^{D/2-\gamma-1} \int_{\mathbb{M}^D} \frac{dk}{k^{2\alpha} [k^2 - p^2/(x(1-x))]^{\beta+\gamma-D/2}}.$$

Now the integral on the right is of the form of eqn (E.1), or

$$\frac{(2\pi)^D}{\mu^{2\omega}} I(\alpha, \beta + \gamma - D/2; 0, p^2/x(1-x)) = i\pi^{D/2} (-)^{\alpha+\beta+\gamma+\omega} \times \left(\frac{p^2}{x(1-x)} \right)^{D-\alpha-\beta-\gamma} \frac{\Gamma(\alpha + \beta + \gamma - D)\Gamma(D/2 - \alpha)}{\Gamma(\beta + \gamma - D/2)\Gamma(D/2)}$$

using eqn (E.5) and ${}_2F_1(a, b, c; 0) = 1$. This gives

$$M(0, \alpha, 0, \beta, \gamma) = (-)^{\alpha+\beta+\gamma+1} \times \frac{\Gamma(\alpha + \beta + \gamma - D)\Gamma(D/2 - \alpha)\Gamma(\alpha + \beta - D/2)\Gamma(\alpha + \gamma - D/2)}{\Gamma(\beta)\Gamma(\gamma)\Gamma(D/2)\Gamma(2\alpha + \beta + \gamma - D)}.$$

Using the above recurrence relations, we can reduce all of the $M(\{\alpha_i\})$ terms to integrals we know, and then to gamma functions. We can also dispose of most of the $N(\{\alpha_i\})$ terms—the only ones left are represented by diagram 5g, which corresponds to terms with $\alpha_{3,4,5} > 0$ and $\alpha_{1,2} \leq 0$.

The latter integrals are not at all easy to evaluate. To do so, we define the quantities a_1, \dots, a_5 to be the Minkowski invariants in the denominator of eqn (3.15), define $a_6 \equiv p^2$, and define $\alpha_6 \equiv 3D/2 - \Sigma$ for consistency with the $\alpha_{1\dots 5}$. Then, we use the nine possible combinations of k and q in (3.17) to express the nine operators $\{-a_i d/da_j : i = 1, 2; j = 3, 4, 5, 6\}$ and $-a_6 d/da_6$, in terms of a_i and $-d/da_i$. The choice of this particular set of nine combinations is not entirely obvious, but is natural in retrospect, given the division, in (3.15), between gluon and quark terms in the denominator. The a_i and $-d/da_i$ lower and raise the coefficients in $N(\{\alpha_i\})$ so that a judicious selection of the operators should be able to generate from a single given integral (say $N(1, 1, 1, 1, 1)$) all the difficult integrals which appear in the calculation. This is exactly what we are able to do.

We take the two operators

$$-a_1 \frac{\partial}{\partial a_3}, \quad \text{and} \quad -a_1 \frac{\partial}{\partial a_6},$$

and express them in terms of a_i and $-\partial/\partial a_i$. Remembering that we wish to act on integrals corresponding to diagram 5g, which is free of gluons, we can set $\partial/\partial \alpha_{1,2} = 0$, which gives

$$\begin{aligned} -a_1 \frac{\partial}{\partial a_3} &= 2D + 2a_6 \frac{\partial}{\partial a_5} + 2a_6 \frac{\partial}{\partial a_4} - 2\alpha_5 - 2\alpha_4 - \alpha_3 \\ -a_1 \frac{\partial}{\partial a_6} &= 2D - a_5 \frac{\partial}{\partial a_4} - a_4 \frac{\partial}{\partial a_5} - a_3 \frac{\partial}{\partial a_6} + a_3 \frac{\partial}{\partial a_5} + a_3 \frac{\partial}{\partial a_4} \\ &\quad - \alpha_5 - \alpha_4 - 2\alpha_3. \end{aligned} \quad (3.21)$$

We can now reexamine eqn (3.18) and construct from the integrand $F(\alpha_1, \dots, \alpha_5) = \prod^6 a_i^{-\alpha_i}$ the identities

$$\begin{aligned} a_1 F(0, 0, \alpha_3, \alpha_4, \alpha_5) &\equiv F(-1, 0, \alpha_3, \alpha_4, \alpha_5) \\ &\equiv \frac{-a_1 [d/da_3 F(0, 0, \alpha_3 - 1, \alpha_4, \alpha_5)]}{(\alpha_3 - 1)} \\ &\equiv \frac{-a_1 [d/da_6 F(0, 0, \alpha_3, \alpha_4, \alpha_5)] m_0^2}{(\frac{3}{2}D - \alpha_3 - \alpha_4 - \alpha_5)}, \end{aligned} \quad (3.22)$$

involving the independent quantities a_i . Using this, eqn (3.21) corresponds to the recurrence relations

$$\begin{aligned} \alpha_3 N(-1, 0, \alpha_3 + 1, \alpha_4, \alpha_5) \\ = \left[2D - \alpha_3 - 2\alpha_4 - 2\alpha_5 - 2\alpha_4 4^+ - 2\alpha_5 5^+ \right] N(0, 0, \alpha_3, \alpha_4, \alpha_5), \end{aligned} \quad (3.23)$$

$$\begin{aligned} (\alpha_3 + \alpha_4 + \alpha_5 - 3D/2) N(-1, 0, \alpha_3, \alpha_4, \alpha_5) &= \left[2\alpha_3 + \alpha_4 + \alpha_5 - 2D \right. \\ &\quad \left. + \alpha_4 4^+ [3^- - 5^-] + \alpha_5 5^+ [3^- - 4^-] + (\alpha_3 + \alpha_4 + \alpha_5 - 3D/2) 3^- \right] \\ &\quad \times N(0, 0, \alpha_3, \alpha_4, \alpha_5). \end{aligned} \quad (3.24)$$

By equating the right hand sides in (3.22), we can generate four simultaneous equations: eqn (3.22) with $\{\alpha_3, \alpha_4, \alpha_5\} = \{2, 1, 1\}$, $\{2, 2, 1\}$ and $\{3, 1, 1\}$; and the identity $I(\omega) \equiv N(1, 1, 1, 1, 1)$. This is a set of very large equations which expand to all the hard integrals we need, plus a host of simpler integrals we can deal with by the methods described earlier in this section. We (or rather, REDUCE) can solve this set of equations to obtain expressions for the relevant hard integrals in terms of $I(\omega)$ and simple integrals. This means that, for this complete calculation, $I(\omega)$ is the *only* two-loop massive integral which must be evaluated.

The value of $I(\omega)$ is needed only at $\omega = 0$. This was determined by Broadhurst in [40], by analytically intensive methods, as

$$I(0) = \pi^2 \log 2 - \frac{3}{2}\zeta(3). \quad (3.25)$$

The techniques we have described in this section can be used to calculate all the diagrams of fig 4. The program which does the calculation is reproduced in appendix C.

3.4 Calculation of d_2 with one massive quark

We are now in a position to calculate the term d_2 in eqn (3.7). To do this we must combine the series for M/m_0 which we calculated in eqn (3.9) and the two-loop mass renormalisation $m_0/m(M)$, taken at the pole mass M . Because the latter ratio involves the renormalised mass, we also need the well known series for coupling constant renormalisation $\alpha_{s,0}/\alpha_s(\mu)$, to one loop. That is, we need

$$\frac{g_0^2}{4\pi} = \left(\frac{\mu^2 e^\gamma}{4\pi} \right)^\omega \alpha_s(\mu) \left[1 - \frac{\alpha_s(\mu)}{\pi} \frac{1}{\omega} \left[\frac{11}{12} C_A - \frac{1}{3} T_F N_F \right] + O(\alpha_s^2(\mu)) \right] \quad (3.26)$$

$$m_0 = m(M) \left[1 + \frac{\alpha_s(M)}{\pi} \frac{1}{\omega} Z_{11} + \left(\frac{\alpha_s(M)}{\pi} \right)^2 \left\{ \frac{1}{\omega^2} Z_{22} + \frac{1}{\omega} Z_{21} \right\} + O(\alpha_s^3(M)) \right] \quad (3.27)$$

$$m_0 = M \left(1 + \sum_{n=1}^{\infty} \left[\frac{g_0^2}{(4\pi)^{D/2} M^{2\omega}} \right]^n C_n \right), \quad (3.28)$$

where the coefficients C_n in (3.28) are given in eqn (3.9). We divide (3.28) by (3.27) and substitute (3.26) to get the required ratio $M/m(M)$. This must be a *finite* function as $\omega \rightarrow 0$, so we adjust the constants Z_{11} , Z_{21} and Z_{22} to remove the poles in ω . Before we can do this, however, we must complete the calculation of the two-loop term C_2 by evaluating $A_2(1)$.

To do this, we express the two-loop diagrams of fig 4 as on-shell integrals in an arbitrary gauge and then use equations (3.19) to (3.24) to reduce these integrals to a single truly hard one, plus products of one-loop integrals, as described in section 3.3 and illustrated in fig 5.

This gives us an expression for $\Sigma(\not{p})|_{2\text{loop}}$, and allows us to define, as we did above for the one-loop case,

$$\begin{aligned} \overline{\Sigma} &= \frac{1}{4} \text{Tr } i(1 + \Delta \not{p})(-i \Sigma|_{2\text{loop}}) \\ &= \Omega[m_0(A_1 - B_1) + \Delta p^2 B_1] + \Omega^2[m_0(A_2 - B_2) + \Delta p^2 B_2], \end{aligned} \quad (3.29)$$

giving

$$A_2(1) = \frac{1}{\Omega} \left(\frac{\overline{\Sigma}}{m_0 \Omega} - A_1(1) \right)_{\Delta=1/m_0}.$$

$$\begin{aligned}
A_{11} &= \frac{-3(5D^3 - 58D^2 + 180D - 152)}{2(3D - 8)(3D - 10)(D - 3)} \\
A_{12} &= \frac{-4(4D^3 - 41D^2 + 122D - 104)}{3(3D - 8)(3D - 10)(D - 3)} \\
A_{13} &= \frac{4(D^2 - 7D + 8)(D - 3)(D - 6)}{(3D - 8)(3D - 10)} \\
A_{21} &= \frac{9D^5 - 84D^4 + 248D^3 - 175D^2 - 226D + 168}{(3D - 8)(3D - 10)(D - 3)^2} \\
A_{22} &= \frac{8(6D^4 - 78D^3 + 355D^2 - 677D + 454)}{3(3D - 8)(3D - 10)(D - 3)^2} \\
A_{23} &= \frac{-8(D^2 - 7D + 8)(D - 3)(D - 6)}{(3D - 8)(3D - 10)} \\
A_{31} &= 0 \\
A_{32} &= \frac{16(D - 2)}{3(3D - 8)(3D - 10)} \\
A_{33} &= 0 \\
A_{41} &= \frac{12(D^3 - 12D^2 + 50D - 68)}{(3D - 8)(3D - 10)(D - 3)(D - 6)} \\
A_{42} &= \frac{-32(D^3 - 9D^2 + 21D - 10)}{3(3D - 8)(3D - 10)(D - 3)(D - 6)} \\
A_{43} &= \frac{8(D^3 - 7D^2 + 6D + 16)(D - 4)}{(3D - 8)(3D - 10)(D - 6)}
\end{aligned}$$

Table 3 Coefficients A_{ij} in eqn (3.30).

After substantial amounts of CPU time, REDUCE gave us

$$A_2(1) = \sum_{i=1}^4 \sum_{j=1}^3 N_i A_{ij} R_j \quad (3.30)$$

where

$$\begin{aligned}
N_1 &= C_A C_F, \quad N_2 = C_F^2, \quad N_3 = C_F N_F, \quad N_4 = C_F, \\
R_1 &= \Gamma^2(\omega), \quad R_2 = \frac{\omega \Gamma^2(-\omega) \Gamma(-4\omega) \Gamma(2\omega) \Gamma(\omega)}{\Gamma(-2\omega) \Gamma(-3\omega)}, \quad R_3 = I(\omega),
\end{aligned}$$

with $C_A = N_C$, for a gauge group $SU(N_C)$, and the coefficients A_{ij} are given in table 3.

The structure R_1 is associated with diagrams 5e, h and i; R_2 with diagrams 5f and j; and R_3 with diagram 5g. The colour factor N_4 is due to a single massive quark in diagram 4d, whilst N_3 results from $N_F - 1$ massless quarks in the same diagram. Note

N_F	d_1	c_1	b_1	d_2	c_2	b_2
3	0.593	0.398	0.790	2.771	0.535	0.883
4	0.640	0.487	0.739	2.992	0.763	0.702
5	0.696	0.613	0.658	3.251	1.120	0.401

Table 4 Coefficients of leading and next-to-leading corrections

that our result for $A_2(1)$ is gauge invariant, in all dimensions D . The consequent gauge invariance of the pole mass of eqn (3.9) provides a strong check on our procedures.

Now we can find d_2 . We use eqns (3.26) to (3.28) to find an expression for $M/m(M)$, adjust the Z_{ij} to cancel the poles, and use our on-shell result (3.30), to obtain the ultraviolet minimal subtractions

$$Z_{11} = -\frac{3}{4}C_F$$

$$Z_{21} = -\frac{97}{192}C_A C_F - \frac{3}{64}C_F^2 + \frac{5}{96}C_F N_F$$

$$Z_{22} = \frac{11}{32}C_A C_F + \frac{9}{32}C_F^2 - \frac{1}{16}C_F N_F,$$

with $T_F = \frac{1}{2}$. Referring back to eqn (2.31) in chapter 2, we see that these agree with the results obtainable from the renormalisation group functions of table 2 or equivalently that we have, in passing, described another calculation of $\gamma_{m,2}$, and obtained results which agree with a much simpler deep-euclidean calculation [19]. We have therefore confirmed that eqn (3.9) is free of infrared singularities by doing a calculation in a general gauge. Our technique is quite unnecessarily powerful for the calculation of these RG coefficients, but the extra complication is essential for the calculation of the coefficients of eqn (3.30).

Our result is gauge invariant and infrared finite, which provide strong checks on our final result. Thus, we have calculated the next-to-leading order correction to the one-loop expression (3.1), as parameterised in (3.7), and find it to be

$$d_2 = \left(\frac{1}{9}\pi^2 \ln 2 - \frac{19}{36}\pi^2 - \frac{1}{6}\zeta(3) + \frac{665}{144}\right)d^2 + \left(\frac{1}{3}\pi^2 + \frac{71}{24}\right)d \quad (3.31)$$

$$\approx -0.031d^2 + 6.248d \quad (3.32)$$

which we derive from the expressions (3.26) to (3.28), and the integral (3.25). In table 4 we give the values of the expansion coefficients for $N_F = 3, 4, 5$. Note that d_2 dominates the next-to-leading corrections.

		α_s	0.25			0.30			0.35		
		L	1	2	3	1	2	3	1	2	3
N_F	M										
5	$M_b =$	4.80	5.89	5.74	5.65	6.08	5.97	5.88	6.27	6.21	6.13
		4.70	5.75	5.61	5.52	5.94	5.83	5.74	6.12	6.06	5.98
		4.60	5.62	5.47	5.38	5.80	5.68	5.59	5.98	5.91	5.83
4	$M_c =$	1.50	1.59	1.48	1.41	1.61	1.49	1.40	1.63	1.50	1.39
		1.45	1.53	1.42	1.35	1.55	1.43	1.34	1.57	1.43	1.32
		1.40	1.47	1.36	1.29	1.49	1.37	1.27	1.50	1.37	1.26
3	$M_s =$	0.55	0.49	0.42	0.36	0.48	0.38	0.30	0.47	0.34	0.23
		0.50	0.44	0.36	0.31	0.43	0.33	0.24	0.41	0.29	0.17
		0.45	0.39	0.31	0.26	0.37	0.27	0.19	0.36	0.23	0.11

Table 5 $m(1 \text{ GeV})$, for L loops, for $\alpha_s(1 \text{ GeV}) = 0.30 \pm 0.05$. These results are also shown in figure 6.

In table 4, it is notable that *all* the coefficients are positive, for $N_F = 3, 4, 5$. Given μ and $\alpha_s(\mu)$, the corrections b_i increase $b(x)$, and so increase $\alpha_s(M)$ in eqn (3.4), for $M < \mu$. Similarly, the c_i increase $c(x)$, and so increase $m(M)$. Finally, the d_i decrease the ratio $m(M)/M$, reducing the current to constituent mass ratio $m(\mu)/M$ by, potentially, a large factor. The extent to which this is true is illustrated in table 5 and fig 6, in which we give the values of this ratio for various values of M and N_F , and for values of the coupling $\alpha_s(1 \text{ GeV})$ varying from 0.25 to 0.35. We obtained these figures by solving eqn (3.4) for $\alpha_s(M)$, for a particular ratio of μ^2/M^2 , and then using that value in eqns 3.6 and 3.7 (using, in the latter, the numerical result given in eqn (3.32)). To illustrate the effects of the higher order corrections, we have shown the results for $L = 1, 2$ and 3 loops, by successively ignoring the terms $\{b_i, c_i, d_i : i \geq L\}$ in each of the ratios in eqn (3.3). In the loop of fig 4d, we can ignore quarks which have a mass greater than M , since they decouple [46] from physical amplitudes at momenta of order M . By this method, we have an expression for the ratio $m(1 \text{ GeV})/M$ which depends on N_F and (implicitly) the number of terms L in the expansions, but which is independent of the RG invariants Λ and \hat{m} , whose values, extracted from experiment, tend to vary widely.

As an aside, we point out that in fig 6 the two- and three-loop corrections are of the same order. We can change this, and attempt to optimise the convergence, by choosing the

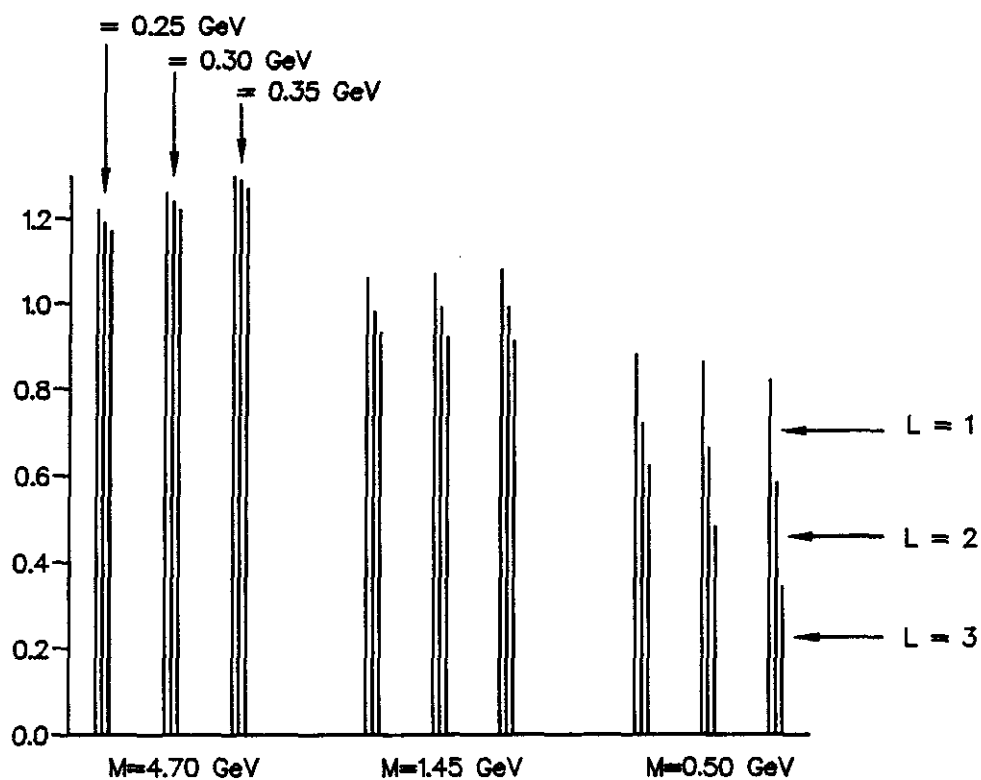


Figure 6 Plot of $m(1 \text{ GeV})/M$, for L loops, for $\alpha(1 \text{ GeV}) = 0.30 \pm 0.05$, and $M = M_b, M_c, M_s$. These data are also shown in table 5.

renormalisation scale μ . We have, from eqn (3.7)

$$\begin{aligned} M/m(M) &= 1 + d_1 \bar{\alpha}(M) + d_2 \bar{\alpha}^2(M) \\ &= 1 + \delta_1 \bar{\alpha}(\mu) + \delta_2 \bar{\alpha}^2(\mu), \end{aligned} \quad (3.33)$$

with $\bar{\alpha}(\mu) = \alpha_s(\mu)/\pi d$. From eqn (3.4), ignoring the next-to-leading corrections,

$$\begin{aligned} \ln \frac{\mu^2}{M^2} &= \int_{\bar{\alpha}(\mu)}^{\bar{\alpha}(M)} \frac{dx}{x^2 + b_1 x^3} \\ &= \left[b_1 (\ln(b_1 x + 1) - \ln x) - 1/x \right]_{\bar{\alpha}(\mu)}^{\bar{\alpha}(M)}. \end{aligned}$$

We want $\bar{\alpha}(M) = \bar{\alpha}(\mu)[1 + \dots]$, so we rearrange this to find

$$\frac{\bar{\alpha}(M)}{\bar{\alpha}(\mu)} = 1 + \bar{\alpha}(M) \left[\ln \frac{\mu^2}{M^2} - b_1^2 (\bar{\alpha}(M) - \bar{\alpha}(\mu)) + b_1 \ln \frac{\bar{\alpha}(M)}{\bar{\alpha}(\mu)} \right] + O(\bar{\alpha}^3).$$

From this, we can show that $\ln \bar{\alpha}(M)/\bar{\alpha}(\mu) = O(\bar{\alpha})$, so that

$$\bar{\alpha}(M) = \bar{\alpha}(\mu)[1 + \bar{\alpha} \ln \mu^2/M^2 + \dots],$$

and, in eqn (3.33), $\delta_1 = d_1$ and $\delta_2 = d_2 + d_1 \ln \mu^2/M^2$ so that the next-to-leading correction to $M/m(M)$ vanishes at a renormalisation scale

$$\mu = M \exp -d_2/2d_1 \approx 0.10M.$$

For both the charm and strange quarks, this is of the order of, or below, the QCD scale Λ , and thus inaccessible to perturbation theory. We cannot, therefore, avoid the sizable corrections which seem to be present, and can only hope that a higher order calculation will show that either the leading correction is accidentally small, or the next-to-leading correction is accidentally large.

The final significance of the figures in table 5 is that they show that perturbation theory might account for rather more of the mass of the strange than has previously been supposed. Conventionally, the disparity between the low current masses of the light quarks, which can be taken to be essentially zero, and their substantial constituent masses, has been accounted for by a non-perturbative term consisting of the non-zero wavefunction VEV $\langle \bar{\psi}\psi \rangle \sim 300 \text{ MeV}$. Since there is no reason to expect any spontaneous breaking of flavour SU(3), this non-perturbative term should have the same value for the strange. This fits in rather well with a current mass (or running mass) of $m_s(1 \text{ GeV}) \approx 150 \text{ MeV}$, giving a constituent mass in the region of 450 MeV.

We feel that our work will support an alternative to this picture, in which much more of the strange mass is due to perturbative effects. From the calculations above, we

can see that a small current or running mass is consistent with a pole mass of the order of twice the size, so that a running mass of 150 MeV would give rise to a perturbative contribution of order 300 MeV. Also, by referring back to eqn (2.55), we can see that the non-perturbative term is suppressed at scales of the order of the pole mass, and might contribute only ~ 200 MeV. This gives, again, a total of ~ 500 MeV, but this time with the strange mass largely perturbative. We have ignored the contributions of higher dimension terms in the OPE, such as $\langle \bar{\psi} G \psi \rangle$, which are suppressed by a factor of M^5 .

As a final point, note that this can refer only to the origin of the strange mass, as the c and b are heavy enough that any non-perturbative contribution is swamped by the large current mass; and the u and d are so light, that the perturbative effects we describe here are still insufficient to let the pole mass compete with the VEV.

3.5 Lighter quark mass corrections

For diagram 4d, we have assumed throughout that the quark loop has one heavy (mass M) quark, and $N_F - 1$ light quarks, going around it. We should check this approximation by explicitly calculating the corrections $\Delta(r)$ to the coefficient K in eqn (3.8) which are due to fermions of mass $M_i = rM$. That is

$$K = K_0 + \sum_{i=1}^{N_F-1} \Delta(M_i/M) \quad (3.34)$$

where the uncorrected

$$K_0 \approx 17.15 - 1.04 N_F$$

comes from eqn (3.31). We can find the $\Delta(r)$ from the finite gauge-invariant difference between the gluon propagators with massive and massless quark loops, $\Pi(M_i^2/Q^2)$. Doing this calculation,

$$\begin{aligned} \Pi(z) &= 2(1-2z)\sqrt{1+4z} \operatorname{arccoth} \sqrt{1+4z} + \ln z + 4z, \\ \Delta(r) &= \frac{1}{24} \int_0^1 dy \left(\frac{4-y^2}{1-y} \right) \Pi(r^2(1-y)/y^2). \end{aligned} \quad (3.35)$$

This was, at the expense of much computer algebra, reduced by Broadhurst [43] to dilogarithms of the form

$$\begin{aligned} L_{\pm}(r) &\equiv \int_0^1 dx \left(\frac{\ln x - \ln r}{x \pm r} \right) = \frac{1}{2} \log^2 r + \sum_{n=1}^{\infty} \frac{(\mp 1)^n}{n^2} (r^n (n \ln r - 1) + 2) \\ &= \ln r \ln \frac{r}{r \pm 1} + \operatorname{Li}_2(\mp 1/r), \quad (r \geq 1) \end{aligned} \quad (3.36)$$

where $\text{Li}_p(x) \equiv \sum_{n=1}^{\infty} x^n/n^p$, for $\{p, x: |x| \leq 1 < p\}$. This gives the result

$$\begin{aligned} \Delta(r) &= \frac{1}{4} \left[\ln^2 r + \frac{1}{6}\pi^2 - (\ln r + \frac{3}{2})r^2 - (1+r)(1+r^3)L_+(r) \right. \\ &\quad \left. - (1-r)(1-r^3)L_-(r) \right] \\ &= \frac{1}{8}\pi^2 r - \frac{3}{4}r^2 + \frac{1}{8}\pi^2 r^3 - \left(\frac{1}{4}\ln^2 r - \frac{13}{24}\ln r + \frac{1}{24}\pi^2 + \frac{151}{288} \right) r^4 \\ &\quad - \sum_{n=3}^{\infty} (2F(n)\ln r + F'(n))r^{2n} \end{aligned}$$

where $F(n) = 3(n-1)/4n(n-2)(2n-1)(2n-3)$, which we have checked numerically. The results above are exact, but have the limiting behaviour

$$\begin{aligned} \Delta(r) &= \frac{1}{4}\ln^2 r + \frac{13}{24}\ln r + \frac{1}{4}\zeta(2) + \frac{151}{288} + O(r^{-2}\ln r) \\ \Delta(r) &= \frac{3}{4}\zeta(2)r + O(r^2), \end{aligned} \tag{3.37}$$

and the value $\Delta(1) = \frac{3}{4}\zeta(2) - \frac{3}{8}$ for $r = 1$. The quantity $\Delta(r)/r$ drops by only 25% between $r = 0$ and $r = 1$, so that, for $r \sim M_s/M_c \sim M_c/M_b \approx 0.3$, we can approximate it by the constant function $\Delta(r)/r \approx 1.04$. Given this, the numerical value of (3.34) can be given as

$$d_2/d^2 = K \approx 16.11 - 1.04 \sum_{i=1}^{N_F-1} (1 - M_i/M)$$

accurate to 0.2%.

3.6 Summary

In this chapter, we described a significant extension to the method of integration by parts, and used it to complete a three loop calculation, in which we found the next-to-next-to-leading order term in the ratio of the $\overline{\text{MS}}$ running mass to the pole mass by combining our new two-loop finite terms with three-loop counterterms of ref [4]. This shows us that a rather large proportion of the strange mass might be generated perturbatively, the origin of which was inadequately explained before. The results for c and b quarks are more conventional, and are shown, with the strange results, in table 5.

k	q	$\parallel \times$	$\widehat{12}$	$\widehat{13}$	$\widehat{14}$	$\widehat{15}$	$\widehat{21}$	$\widehat{23}$	$\widehat{24}$	$\widehat{25}$	$\widehat{31}$	$\widehat{32}$	$\widehat{34}$	$\widehat{35}$	$\widehat{41}$	$\widehat{42}$	$\widehat{43}$	$\widehat{45}$	$\widehat{51}$	$\widehat{52}$	$\widehat{53}$	$\widehat{54}$
k_1	k_1	$D - 2\alpha_1 - \alpha_3 - \alpha_5$		-	-													+				
	k_2	$\alpha_3 - \alpha_5$						+		-	+			+	+				-		-	
	p	$\alpha_1 - \alpha_3$		+	+					+	-			-				-				$-2\alpha_3 3^+ - 2\alpha_5 5^+$
k_2	k_1	$\alpha_4 - \alpha_5$			+	-						+				+	+		-	-		
	k_2	$D - 2\alpha_2 - \alpha_4 - \alpha_5$							-	-				+								
	p	$\alpha_2 - \alpha_4$				+			+	+				-		-	-					$-2\alpha_4 4^+ - 2\alpha_5 5^+$
p	k_1	$\alpha_4 - \alpha_5$		-2	-2								+	+				+			-	$+(3D/2 - \Sigma)(1^- - 3^-)$
	k_2	$\alpha_3 - \alpha_5$							-2	-2				+			+	+		-		$+(3D/2 - \Sigma)(2^- - 4^-)$
	p	$2\alpha_1 + 2\alpha_2 + \alpha_3$		+	+				+	+				-				-				
		$+\alpha_4 + 2\alpha_5 - 2D$																				

Table 6 Recurrence relations obtained from eqn (3.17) for different values of k and q , operating on the integral $N(\alpha_1, \dots, \alpha_5)$ of eqn (3.15). Note that of these nine recurrence relations only eight are independent, since $\text{line1} + \text{line5} + \text{line9} = 0$. In the heading, the symbol $\widehat{23}$, for example, represents the pair of lowering and raising operators $\alpha_3 2^- 3^+$, and we have defined $\Sigma = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5$. Thus, taking $k = q = k_1$ as an example, we have $0 = (D - 2\alpha_1 - \alpha_3 - \alpha_5)N(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) - \alpha_3 N(\alpha_1 - 1, \alpha_3 + 1) - \alpha_5 N(\alpha_1 - 1, \alpha_5 + 1) + \alpha_5 N(\alpha_4 - 1, \alpha_5 + 1)$.

k	q	$\parallel \times$	$\widehat{12}$	$\widehat{13}$	$\widehat{14}$	$\widehat{15}$	$\widehat{21}$	$\widehat{23}$	$\widehat{24}$	$\widehat{25}$	$\widehat{31}$	$\widehat{32}$	$\widehat{34}$	$\widehat{35}$	$\widehat{41}$	$\widehat{42}$	$\widehat{43}$	$\widehat{45}$	$\widehat{51}$	$\widehat{52}$	$\widehat{53}$	$\widehat{54}$
k_1	k_1	$D - 2\alpha_1 - \alpha_2 - \alpha_4$	-	-								+										
	k_2	$\alpha_2 - \alpha_1$	-	-		+		+			-	+									-	
	p	$\alpha_1 - \alpha_4$	+	+								-			-	-				+		$-2\alpha_4 4^+$
k_2	k_1	$\alpha_2 - \alpha_3$	+	-				+		+		-	-					-				
	k_2	$D - \alpha_2 - 2\alpha_3 - \alpha_5$	+									-	-									
	p	$\alpha_3 - \alpha_5$	-									+	+		+					-	-	$-2\alpha_5 5^+$
p	k_1			-2	-					+				-								$+(3D/2 - \Sigma)(1^- - 4^-)$
	k_2			-				+				-	-2									$+(3D/2 - \Sigma)(3^- - 5^-)$
	p	$2\alpha_1 + 2\alpha_2 + 2\alpha_3$		+										+								
		$+\alpha_4 + \alpha_5 - 2D$																				

Table 7 Recurrence relations obtained from eqn (3.17) for different values of k and q , operating on the integral $M(\alpha_1, \dots, \alpha_5)$ of eqn (3.16). The notation is as in table 6.

Chapter 4

Wavefunction renormalisation

In section 2.6, we described how we can obtain an expression for Z_2 in terms of A_i , B_i and their derivatives at $\not{p} = M$, and displayed this expression in eqn (2.56). Having seen in the previous chapter how we can evaluate, on shell, the complicated massive integrals which appear, we will be in a position to continue the calculation when we have re-expressed this result in terms of A_i and B_i at $\not{p} = m_0$.

What we will find is that the wavefunction renormalisation constant is rather simpler than we might expect, and that it is also gauge invariant to two loops. This raises the possibilities (i) that the simplicity of Z_2 is not ‘accidental’, and there is some undiscovered principle which would allow us to derive this, and (ii) that Z_2 is gauge invariant to all orders, although we can see no physical reason why this should be so.

This work involves an extension to the techniques of the last section, in a complicated and lengthy series of calculations. We have considerable confidence in our result, because the gauge cancellations in Z_2 are so intricate and extensive that they are impossibly unlikely to have happened by chance.

From our result we are also able to extract an important anomalous dimension of the EFT of the static quark.

The material described in this chapter was first published by D J Broadhurst, K Schilcher and myself in ref [47].

4.1 On-shell expression for Z_2

To recap, we saw in eqn (2.56) how to derive for Z_2 the expression

$$iZ_2^{-1} = (1 - B) \left(1 - 2 \frac{A' - B'}{1 + A - B} - 2 \frac{B'}{1 - B} \right) \bigg|_{\not{p}^2 = M^2} \quad (4.1)$$

where $A' = \partial A / \partial \ln p^2$. We must evaluate A_i and B_i at $p^2 = M^2$ by making a Taylor expansion

$$\begin{aligned} A(\ln p^2 = \ln M^2) &= A(\ln m_0^2) + (\ln M^2 - \ln m_0^2) \left. \frac{\partial A}{\partial \ln p^2} \right|_{p^2=m_0^2} \\ &\quad + \frac{1}{2} (\ln M^2 - \ln m_0^2)^2 \left. \frac{\partial^2 A}{\partial (\ln p^2)^2} \right|_{p^2=m_0^2}. \end{aligned}$$

We know from eqn (2.52) that $m_0 = Z_M M$, with

$$\begin{aligned} Z_M^{-1} - 1 &= \sum_n \Omega^n [A_n(\ln M^2) + (Z_M^{-1} - 1) B_n(\ln M^2)] \\ &= \sum_n \Omega^n \left[A_n(\ln m_0^2) + \ln Z_M^{-2} \left. \frac{\partial A_n}{\partial \ln p^2} \right|_{p^2=m_0^2} \right. \\ &\quad \left. + (Z_M^{-1} - 1) \left(B_n(\ln m_0^2) + \ln Z_M^{-2} \left. \frac{\partial B_n}{\partial \ln p^2} \right|_{p^2=m_0^2} \right) \right] \end{aligned} \quad (4.2)$$

from which we can see, after a little manipulation, that

$$\ln Z_M^{-2} = 2\Omega A_1(1 + 2\Omega A_1) = O(\Omega)$$

(where $A_i^{(n)} \equiv A_i^{(n)}(\ln p^2 = \ln m_0^2)$ here and below), and so

$$Z_M^{-1} = 1 + \Omega A_1 + O(\Omega^2),$$

which, on substitution back into eqn (4.2), gives

$$Z_M^{-1} - 1 = \Omega A_1 + \Omega^2 [A_2 + A_1(2A_1' + B_1)] + O(\Omega^3).$$

We can now put all this together, remembering that $A_i^{(n)}$ are of order Ω , to obtain

$$A(p^2 = M^2) = A(p^2 = m_0^2) + 2\Omega A_1 A'(\ln p^2 = \ln m_0^2) + O(\Omega^3)$$

$$B(p^2 = M^2) = B(p^2 = m_0^2) + 2\Omega A_1 B'(\ln p^2 = \ln m_0^2) + O(\Omega^3).$$

Pressing on, we remember that $\Omega \propto (p^2)^{-\omega}$, and obtain

$$A'(\ln p^2) = \Omega(A_1' - \omega A_1) + \Omega^2(A_2' - 2\omega A_2)$$

$$A''(\ln p^2) = \Omega(A_1'' - 2\omega A_1' + \omega^2 A_1) + (\Omega^2(A_2'' - 4\omega A_2' + 4\omega^2 A_2))$$

(the expressions for B' and B'' are the same, with $A_i \rightarrow B_i$). If we expand the A' and B' in a Taylor series, $A'(\ln M^2) = A'(\ln m_0^2) + (Z_M^{-2} - 1)A''(\ln m_0^2)$, we get

$$A'(\ln M^2) = A'(\ln m_0^2) + 2\Omega^2 A_1(A_1'' - 2\omega A_1' + \omega^2 A_1) + (\Omega^3)$$

$$B'(\ln M^2) = B'(\ln m_0^2) + 2\Omega^2 A_1(B_1'' - 2\omega B_1' + \omega^2 B_1) + (\Omega^3).$$

We now have $A(p^2 = M^2)$ and $B(p^2 = M^2)$, and thus $Z_2(A, B)$, expressed in terms of $A_i(p^2 = m_0^2)$ and $B_i(p^2 = m_0^2)$ and their derivatives. Putting all these expansions together, using REDUCE again, we end up with the expansion

$$Z_2 = 1 + \sum_{n=1}^{\infty} \left[\frac{g_0^2}{(4\pi)^{D/2} M^{2\omega}} \right]^n F_n \quad (4.3)$$

with

$$\begin{aligned} F_1 &= -2\omega A_1 - 2A_1' + B_1, \\ F_2 &= 4\omega^2 A_1^2 + 2\omega A_1'^2 + 12\omega A_1 A_1' - 6\omega A_1 B_1 - 4\omega A_2 + 6A_1 A_1' + 4A_1 A_1'' \\ &\quad - 4A_1 B_1' + 4A_1'^2 - 4A_1' B_1 + B_1^2 - 2A_2' + B_2. \end{aligned}$$

We worked out the values of A_1 , B_1 and A_1' in section 3.2, where we also worked out an expansion, in eqn (3.9), for the pole mass. Substituting into F_1 and C_1 , we find that

$$F_1 = C_1 = -C_F \frac{D-1}{D-3} \Gamma(\omega),$$

so that we discover that Z_M and Z_2 are both gauge invariant, and equal to each other at the one-loop level. For QED, we know from the dimensionally regulated version of the Johnson-Zumino identity [37]

$$\frac{d \ln Z_2}{da_0} = i(2\pi)^{-D} e_0^2 \int \frac{d^D k}{k^4} = 0$$

that Z_2^{QED} must be gauge invariant to all orders. There is no extension of this to QCD, however, and certainly no obvious reason why Z_2 and Z_M should be equal. When we finish the two-loop calculation in section 4.2, we will see that this equality is in fact something of a coincidence, and that the only remnant of it at that order is an unexpectedly simple relationship between F_2 and C_2 .

The on-shell A_i and B_i may be extracted from $\Sigma(p^2)$ and so, before we go on to show how the methods of chapter 3 must be extended to deal with the two-loop diagrams, we shall show how this extraction is done.

We saw in eqn (3.29) how A_i and B_i can be extracted from the quantity $\bar{\Sigma}(p^2)$. If we differentiate this, we find

$$\begin{aligned} \sigma \equiv \frac{\partial \bar{\Sigma}}{\partial \ln p^2} &= -\Omega^2 m_0 (m_0 (2\omega - 1) \Delta B_2 - m_0 \Delta B_2' + 2\omega (A_2 - B_2) - A_2' - B_2') \\ &\quad - \Omega m_0 (m_0 (\omega - 1) \Delta B_1 - m_0 \Delta B_1' + \omega (A_1 - B_1) - A_1' + B_1') \end{aligned}$$

which, when we have defined $\bar{\sigma} \equiv [\sigma/m_0\Omega - (\sigma/m_0\Omega)|_{\Omega=0}]/\Omega$, gives us

$$\begin{aligned} A_2' &= \bar{\sigma} \Big|_{\Delta=1/m_0} - B_2 + 2\omega A_2 \\ B_2' &= A_2' - \bar{\sigma} \Big|_{\Delta=0} - 2\omega (A_2 - B_2). \end{aligned}$$

Similarly, using

$$\sigma' \equiv \frac{1}{m_0 \Omega} \frac{\partial^2 \bar{\Sigma}}{\partial (\ln p^2)^2} \Big|_{\Omega=0},$$

we can obtain the one-loop second derivatives (which are all we need) via

$$A_1'' = \sigma' \Big|_{\Delta=1/m_0} - B_1 - 2B_1' + 2\omega(A_1' + B_1) - \omega^2 A_1$$

$$B_1'' = A_1'' - \sigma' \Big|_{\Delta=0} - 2\omega(A_1' - B_1') + \omega^2(A_1 - B_1).$$

Collecting all the one-loop terms, we have, therefore,

$$A_1(1) = C_F \frac{D-1}{D-3} \Gamma(\omega) \quad (3.13)$$

$$A_1'(1) = \frac{1}{2} C_F \left(D-1 - \frac{a_0}{D-3} \right) \Gamma(\omega). \quad (3.14)$$

$$A_1''(1) = C_F \left(\frac{(D-6)((D-1)(D-4) - 2a_0)}{4(D-3)} \right) \Gamma(\omega) \quad (4.4)$$

$$B_1(1) = -C_F \frac{a_0}{D-3} \Gamma(\omega) \quad (3.13)$$

$$B_1'(1) = -C_F \left(\frac{(D-2)a_0}{2(D-3)} \right) \Gamma(\omega). \quad (4.5)$$

We must use REDUCE again to differentiate $\bar{\Sigma}$. This must be done carefully, not only because the p -dependence of $\bar{\Sigma}$ is complicated, but because a bad method could be very expensive of computer time. We express $\bar{\Sigma}$ in terms of the invariants a_1, \dots, a_5 (off shell) in eqn (3.15). That is, we make the replacements

$$\begin{aligned} k_1^2 &= a_1, & p \cdot k_1 &= \frac{1}{2}(m_0^2 - p^2 + a_3(p^2) - a_1), \\ k_2^2 &= a_2, & p \cdot k_2 &= \frac{1}{2}(m_0^2 - p^2 + a_4(p^2) - a_2), \\ k_1 \cdot k_2 &= \frac{1}{2}(p^2 - m_0^2 + a_5(p^2) - a_3(p^2) - a_4(p^2)) \end{aligned}$$

and use the expressions

$$\begin{aligned} p \cdot \frac{\partial a_3}{\partial p} &= \frac{1}{2}(a_3(p^2) - a_1 + p^2 + m_0^2), & p \cdot \frac{\partial a_4}{\partial p} &= \frac{1}{2}(a_4(p^2) - a_2 + p^2 + m_0^2) \\ p \cdot \frac{\partial a_5}{\partial p} &= \frac{1}{2}(a_3(p^2) + a_4(p^2) - a_1 - a_2) + m_0^2 \end{aligned}$$

to do the differentiations and obtain $\partial \bar{\Sigma} / \partial \ln p^2$ for all of the one- and two-loop diagrams, and $\partial^2 \bar{\Sigma} / \partial (\ln p^2)^2$ for the one-loop diagram alone. Once this is done, we can use the program¹ minnie.rd3 reproduced in appendix C, to express the results in terms of gamma functions, for which we will need to extend slightly the method of integration by parts which we described in the last chapter. After that, we use the expressions we have derived to extract the terms we need from the results.

¹Note that the integrals in this program are not precisely the same as the integrals described in the text, but are related (essentially through Wick rotations) by a factor of $(-1)^{\Sigma a_i + 1}$.

4.2 Integration by parts

The techniques described in section 3.3 are almost sufficient as they stand to evaluate the on-shell expression for $\partial^{(n)}\bar{\Sigma}/\partial(\ln p^2)^{(n)}$. It is the presence of the second derivatives of $\bar{\Sigma}$ which confound them, when they produce integrals of the form $N(-2, \alpha_2, \dots, \alpha_5)$ (see section 3.3). To deal with these, we must do a little extra work, and apply $-a_1 d/da_6$ to $N(-1, 0, \alpha_3, \alpha_4, \alpha_5)$, and solve the resulting expression for $N(-2, 0, \alpha_3, \alpha_4, \alpha_5)$ —it is this extended version of the program which is reproduced in appendix C. With this addition to the list of integrals we know, we can calculate the two-loop term F_2 of eqn (4.3). We present it in the combination

$$F_2 - (1 + \frac{D}{4})C_2 \equiv \sum_{i=1}^4 \sum_{j=1}^2 N_i F_{ij} R_j, \quad (4.6)$$

using the same notation as was used for eqn (3.30), with the coefficients F_{ij} given in table 8. Notice that the sum over j runs only to $j = 2$ —there are no terms proportional to $I(0)$ in this combination. This (rather contrived) cancellation of $I(0)$ is probably the only remnant of the evidence for a simple relation between Z_M and Z_2 , although the fact that we can reasonably easily construct such a quantity free of the only truly hard integral in the calculation suggests that some further explanation should be possible.

This (relatively) simple form is due to detailed cancellations between terms in diagrams 4b and d with three intermediate fermions. We can find no sense in which it is ‘obvious’, and it is such an unlikely thing to happen by chance that one finds oneself speculating that in any L loop calculation we could find a linear combination of F_L and C_L which is free of contributions from diagrams with the maximum number of intermediate fermions, $2L - 1$.

The terms in table 8 are also notably gauge invariant, as is the full expression for F_2 , which we evaluated for all D , and for all a_0 . The expression itself is of the general form of eqn (4.6), but is more bulky and entirely uninformative. This cancellation of the gauge parameter is even more remarkable than the cancellation above, as it involves terms up to a_0^2 , and terms involving several of the structures in eqn (3.30). Although we have said that we cannot keep track of UV and IR divergences in dimensional regularisation, we can compare our results with earlier calculations which take no account of IR terms [19,48], and see that all the singular terms in C_2 must be UV divergences, and that F_2 must have gauge dependent contributions from both IR and UV singularities, which *cancel*.

Although our results have no contributions from the four-gluon coupling, it would be surprising if the intricate cancellations we have uncovered here were not derivable from some principle which also applied to that coupling, as well as to higher orders in the perturbative

$$\begin{aligned}
F_{11} &= -\frac{3D^5 - 61D^4 + 469D^3 - 1679D^2 + 2756D - 1648}{8(D-3)^2(D-5)^2} \\
F_{12} &= -\frac{2D^5 - 29D^4 + 148D^3 - 321D^2 + 268D - 60}{3(3D-10)(D-3)^2(D-5)} \\
F_{21} &= \frac{(2D^3 - 29D^2 + 134D - 187)(D-1)(D-4)}{4(D-3)^2(D-5)^2} \\
F_{22} &= \frac{2(2D^3 - 21D^2 + 63D - 50)(2D-7)(D-1)}{3(3D-10)(D-3)^2(D-5)} \\
F_{31} &= 0 \\
F_{32} &= \frac{4(D-2)}{3(3D-10)} \\
F_{41} &= -\frac{2(D^2 - 8D + 11)(D-4)}{(D-2)(D-3)(D-5)(D-7)} \\
F_{42} &= -\frac{4(D-2)}{3(3D-10)}
\end{aligned}$$

Table 8 Coefficients F_{ij} in eqn (4.6)

series. This explanation might come from the pole in the on-shell six-point amplitude, which relates two on-shell four-point amplitudes and Z_2 , so that one might be able to use the gauge invariance of the on-shell amplitudes to explain the gauge invariance of Z_2 . One would then have to explain why there is no hint of this invariance in other regulation schemes, but some progress would have been made.

Whatever the explanation, we may speculate that the wavefunction renormalisation constant Z_2 is gauge invariant to all orders in non-Abelian gauge theories.

4.3 Laurent expansions in QCD and QED

We shall now continue our calculation by exhibiting the results of a Laurent expansion, in $1/\omega$, of Z_2 and Z_M . This is an operation of nightmarish complexity, involving perturbative expansions in $\Omega(p^2)$ about both $p^2 = m_0^2$ and $p^2 = M^2$, and Taylor expansions to allow us to move from the on-shell ($p^2 = m_0^2$) results we can calculate to the expressions at $p^2 = M^2$ which we need. The REDUCE program which does the calculation is reproduced in appendix C on page 110.

The calculation was performed in the $\overline{\text{MS}}$ scheme, but on-shell renormalisation is of more use in QED, in which we want to relate the coupling to the experimentally measured value $\simeq 1/137$. Therefore we will go on to re-exhibit our results as on-shell renormalised ones in QED. The difference between the two renormalisation methods is rather subtle, in fact, as the renormalised couplings are equal as $\omega \rightarrow 0$.

$\overline{\text{MS}}$ renormalisation

The program in appx C obtains an expression for Z_2 (called *Zf* in the program) in terms of $A(M^2)$ and $B(M^2)$ (called *AM* and *BM*) which are expanded, in turn, in $\Omega(M^2)$. The coefficients of that Ω -expansion are then Taylor-expanded so that we finally reach an expression for Z_2 in terms of the A_1, A'_1, \dots (called *a1m*, *a1mp*...) which we can calculate. The result is the expression for Z_2 of eqn (4.3) (the F_i are called D_i), to which we can add the expansion for Z_M of eqn (3.9). We extract these coefficients from $\overline{\Sigma}$, and prepare to renormalise them.

QCD coupling constant renormalisation is not trivial, but it is well known, and we can simply use the one-loop expression of eqn (3.26). As we did in chapter 3, we can plug this expression into the unrenormalised expressions (4.1) and (4.2) we extracted above, and go about extracting the coefficients of $\{\omega^i: i \geq -2\}$. Note that we must retain the $O(\omega^1)$ term in the one-loop expressions, as it can feed through to the two-loop ones if it is multiplied by an $O(\omega^{-1})$ term in another expansion.

Thus, in full generality, the expansions are as follows (where we have written, for convenience, $\alpha_s^{\text{MS}}(\mu = M) = \alpha_{s,M}^{\text{MS}}$)

$$Z_M = 1 + \left(C_1^1 \omega^{-1} + C_1^0 + C_1^{-1} \omega^1 + O(\omega^2) \right) \frac{\alpha_{s,M}^{\text{MS}}}{\pi} \\ + \left(C_2^2 \omega^{-2} + C_2^1 \omega^{-1} + C_2^0 + O(\omega) \right) \left(\frac{\alpha_{s,M}^{\text{MS}}}{\pi} \right)^2 + O \left(\frac{\alpha_{s,M}^{\text{MS}}}{\pi} \right)^3 \quad (4.7)$$

$$Z_2 = 1 + \left(D_1^1 \omega^{-1} + D_1^0 + D_1^{-1} \omega^1 + O(\omega^2) \right) \frac{\alpha_{s,M}^{\text{MS}}}{\pi} \\ + \left(D_2^2 \omega^{-2} + D_2^1 \omega^{-1} + D_2^0 + O(\omega) \right) \left(\frac{\alpha_{s,M}^{\text{MS}}}{\pi} \right)^2 + O \left(\frac{\alpha_{s,M}^{\text{MS}}}{\pi} \right)^3 \quad (4.8)$$

with

$$C_1^{-1} = -C_F \left(\frac{3}{8} \zeta(2) + 2 \right) \quad C_1^0 = -C_F \quad C_1^1 = -\frac{3}{4} C_F \\ C_2^0 = C_A C_F \left(-\frac{1}{4} I(0) + \frac{1}{2} \zeta(2) - \frac{1111}{384} \right) + C_F^2 \left(\frac{1}{2} I(0) - \frac{51}{32} \zeta(2) + \frac{199}{128} \right) \\ + C_F N_F T_F \left(\frac{1}{2} \zeta(2) + \frac{71}{96} \right) + C_F \left(-\frac{3}{2} \zeta(2) T_F + \frac{3}{4} T_F \right) \quad (4.9) \\ C_2^1 = -\frac{97}{192} C_A C_F + \frac{45}{64} C_F^2 + \frac{5}{48} C_F N_F T_F \\ C_2^2 = \frac{11}{32} C_A C_F + \frac{9}{32} C_F^2 - \frac{1}{8} C_F N_F T_F$$

and

$$D_1^{-1} = -C_F \left(\frac{3}{8} \zeta(2) + 2 \right) \quad D_1^0 = -C_F \quad D_1^1 = -\frac{3}{4} C_F \\ D_2^0 = C_A C_F \left(-\frac{1}{2} I(0) + \frac{15}{8} \zeta(2) - \frac{1705}{384} \right) + C_F^2 \left(I(0) - \frac{147}{32} \zeta(2) + \frac{433}{128} \right) \\ + C_F N_F T_F \left(\frac{1}{2} \zeta(2) + \frac{113}{96} \right) + C_F \left(-\frac{19}{8} \zeta(2) T_F + \frac{19}{9} T_F \right) \quad (4.10) \\ D_2^1 = -\frac{127}{192} C_A C_F + \frac{51}{64} C_F^2 + \frac{11}{48} C_F N_F T_F - \frac{1}{6} C_F T_F \\ D_2^2 = \frac{11}{32} C_A C_F + \frac{9}{32} C_F^2 - \frac{1}{8} C_F N_F T_F + \frac{1}{8} C_F T_F$$

On-shell renormalisation

In QED, one can renormalise the coupling, and obtain $Z_3 = e_R^2/e_0^2$, by calculating the wavefunction renormalisation of the photon at $q^2 = 0$. The non-Abelian source of the gluon prevents this in QCD, because the three-gluon coupling (and the fact of massless gluons) complicates the ‘Coulomb interaction’ below the $\bar{q}q$ threshold—the gluon loop in that interaction produces a term proportional to $\int dk/k^4$ which disappears in dimensional regulation.

A two loop expression for the photon self-energy $\Pi_{\mu\nu}(q)$ can be obtained fairly easily—we can then differentiate this with $\partial^2/\partial q_\mu \partial q_\nu$ and set $q = 0$. This was done (by Broadhurst) in [47], and gives us bubble diagrams, which evaluate to $\Pi(0)$ times a constant

tensor, and from this we can find the on-shell renormalisation constant $Z_3 = 1/(1 + \Pi(0))$. This Z_3 is not minimal, and contains, in the finite terms, much more information than there is in the Z_3 produced by $\overline{\text{MS}}$ renormalisation. This is partly because the e_R to which the e_0 is renormalised is the measured electron charge, in the same way that the pole mass M in QED is the measured electron mass.

As mentioned above, we have *not* used on-shell renormalisation to do any of the calculations in this thesis, although the method was extensively used in our [47] (see appendix B). However, we can take advantage of the relation

$$\begin{aligned}\alpha_M &\equiv \frac{e_R^2}{4\pi} \left(\frac{4\pi}{M^2 e^\gamma} \right)^\omega \\ &= \alpha_s^{\text{MS}}(\mu = M) + O(\alpha_M^2 \omega \zeta(2))\end{aligned}$$

to convert our $\overline{\text{MS}}$ results into the corresponding (and, for QED, rather more applicable) results obtained from on-shell renormalisation. In this expression, the α_M on the left is the on-shell coupling introduced in [47], and is *not* a running coupling, and that on the right is from $\overline{\text{MS}}$.

For QED with one fermion, we can instruct our program to use the on-shell coupling renormalisation constant (that is, the α_M above), and then substitute $C_A = 0$, $T_F = C_F = N_F = 1$, as described in appendix D.4, and find the following simple expressions for the on-shell electron mass and photon wavefunction renormalisation constants.

With

$$\begin{aligned}Z_M &= 1 + \left(\overline{C}_1^1 \omega^{-1} + \overline{C}_1^0 + \overline{C}_1^{-1} \omega^1 + O(\omega^2) \right) \frac{\alpha_M}{\pi} \\ &\quad + \left(\overline{C}_2^2 \omega^{-2} + \overline{C}_2^1 \omega^{-1} + \overline{C}_2^0 + O(\omega) \right) \left(\frac{\alpha_M}{\pi} \right)^2 + O \left(\frac{\alpha_M}{\pi} \right)^3,\end{aligned}$$

and a similar expression for Z_2 , we have

$$\begin{aligned}\overline{C}_1^{-1} &= -\left(\frac{3}{8}\zeta(2) + 2\right) & \overline{C}_1^0 &= -1 & \overline{C}_1^1 &= -\frac{3}{4} \\ \overline{C}_2^0 &= \frac{1}{2}I(0) - \frac{87}{32}\zeta(2) + \frac{1169}{384} & \overline{C}_2^1 &= \frac{155}{192} & \overline{C}_2^2 &= \frac{5}{32}\end{aligned}\tag{4.11}$$

and

$$\begin{aligned}\overline{D}_1^{-1} &= -\left(\frac{3}{8}\zeta(2) + 2\right) & \overline{D}_1^0 &= -1 & \overline{D}_1^1 &= -\frac{3}{4} \\ \overline{D}_2^0 &= I(0) - \frac{211}{32}\zeta(2) + \frac{7685}{1152} & \overline{D}_2^1 &= \frac{55}{64} & \overline{D}_2^2 &= \frac{9}{32}.\end{aligned}\tag{4.12}$$

The numerical values of the second-order finite parts are $\overline{C}_2^0 = 1.09$ and $\overline{D}_2^0 = 0.86$ —relatively small coefficients which indicate, yet again, fine cancellations. These on-shell results are subtly, but importantly, different from the corresponding results after $\overline{\text{MS}}$ renormalisation: the only differences are in the coefficients of $\zeta(2)$ in C_2^0 and D_2^0 , which are $-\frac{83}{32}$ and $-\frac{207}{32}$ respectively.

4.4 Wavefunction renormalisation and effective field theory

In the work we have described up to now, we have been dealing with one heavy quark, and $N_L = N_F - 1$ light quarks. Before we go on to look at the effects of giving these light quarks non-zero masses, we shall describe the effects of giving the heavy quark, instead, an *infinite* mass. This corresponds to the effective field theory (EFT) described in section 2.4. This EFT is relevant to our work, in principle, as it provides a starting point from which to approach the perturbative, and finite, heavy quark mass shell which is one of our main concerns.

The heavy quark is unaffected by coupling constant renormalisation, so we must discard its contribution from the wavefunction renormalisation constant. We do this by setting $N_F = N_L + 1$ in eqn (4.8) and then discarding the terms proportional to simply $C_F T_F$. We will denote the remainder Z_2^L .

We may now connect our calculations to the EFT through the pair of wavefunction anomalous dimensions

$$\gamma_F = \frac{d \ln Z_2^{\text{MS}}(\mu)}{d \ln \mu}$$

$$\tilde{\gamma}_F = \frac{d \ln \tilde{Z}_2^{\text{MS}}(\mu)}{d \ln \mu},$$

where the tilde denotes the EFT, and both constants are obtained through $\overline{\text{MS}}$ renormalisation.

Under the adiabatic hypothesis which is the input to the LSZ reduction formula, the bare field is related to the field of the incoming particles through the *on-shell* renormalisation constant Z_2 , by

$$\lim_{x_0 \rightarrow -\infty} \psi_0(x) = Z_2^{1/2} \psi_{\text{in}}(x).$$

Furthermore, the renormalised field is related to the bare field, through the $\overline{\text{MS}}$ constant Z_2^{MS} , by eqn (2.10)

$$\psi_0(x) = (Z_2^{\text{MS}})^{1/2} \psi_r(x).$$

Now, the Green's functions are defined in terms of the renormalised fields, and S-matrix elements are, through the definition

$$\langle \psi_{f,\text{out}} | \psi_{i,\text{in}} \rangle = \langle \psi_{f,\text{in}} | S | \psi_{i,\text{in}} \rangle,$$

in terms of the asymptotic fields. The latter are therefore related to the former by a factor $(Z_2^{\text{MS}}/Z_2)^{1/2}$ for each of N_F external heavy fermions. The same is true for the EFT

(with tildes on), although we have some additional information: because on-shell diagrams in EFT have no mass scale (because the only quarks in the theory have either zero or infinite mass), we must have $\tilde{Z}_2 = 1$, on shell.

Physical S -matrix elements of the two theories can differ by no more than radiative corrections which vanish in the infinite mass limit of QCD, when $\alpha(M) \rightarrow 0$. Since the $\overline{\text{MS}}$ -renormalised Green's functions in both theories are constructed to be finite, the factor relating them,

$$\frac{S/\tilde{S}}{\Gamma/\tilde{\Gamma}} = \left(\frac{Z_2^{\text{MS}}}{Z_2^{\text{L}}} \frac{1}{\tilde{Z}_2^{\text{MS}}} \right)^{-N_{\text{B}}/2} = (\text{finite}), \quad (4.13)$$

must be finite also. We can calculate this ratio. Specifically,

$$R(\mu) \equiv \frac{1}{Z_2^{\text{L}}} \frac{Z_2^{\text{MS}}}{\tilde{Z}_2^{\text{MS}}},$$

and the ratio of $\overline{\text{MS}}$ wavefunction renormalisation constants in the two theories is

$$\begin{aligned} r(\mu) &\equiv \frac{Z_2^{\text{MS}}}{\tilde{Z}_2^{\text{MS}}} \\ &= 1 - 3C_{\text{F}} \frac{\tilde{a}}{\omega} + C_{\text{F}} \left(\frac{\tilde{a}}{\omega} \right)^2 \left[\frac{11}{2}C_{\text{A}} + \frac{9}{2}C_{\text{F}} - 2T_{\text{F}}N_{\text{L}} \right. \\ &\quad \left. - \left(\frac{127}{12}C_{\text{A}} - \frac{3}{4}C_{\text{F}} - \frac{11}{3}T_{\text{F}}N_{\text{L}} \right) \omega \right] + O(\tilde{a}^3). \end{aligned}$$

where we have defined $\tilde{a} = \alpha_{\text{s}}/4\pi$, and where the subtractions in the latter expression are the minimal ones necessary to make the complete expression for $R(\mu)$ finite.

Using for the beta function the expression $\beta(\alpha_{\text{s}}) = -2\omega + (\frac{2}{3}T_{\text{F}}N_{\text{F}} - \frac{11}{6}C_{\text{A}})(\alpha_{\text{s}}/\pi)$, we can now differentiate this ratio to find the difference

$$\gamma_{\text{F}} - \tilde{\gamma}_{\text{F}} = \frac{3C_{\text{F}}\alpha_{\text{s}}(\mu)}{2\pi} + \left(\frac{127}{12}C_{\text{A}} - \frac{3}{4}C_{\text{F}} - \frac{11}{3}T_{\text{F}}N_{\text{L}} \right) \frac{C_{\text{F}}\alpha_{\text{s}}^2(\mu)}{4\pi^2} + O(\bar{\alpha}^3). \quad (4.14)$$

The QCD wavefunction anomalous dimension is known [49], so that we can derive from the above result the wavefunction anomalous dimension in this EFT [47],

$$\tilde{\gamma}_{\text{F}} = \frac{(a-3)C_{\text{F}}\alpha_{\text{s}}(\mu)}{2\pi} + \left[\left(\frac{a^2}{32} + \frac{a}{4} - \frac{179}{96} \right) C_{\text{A}} + \frac{2}{3}T_{\text{F}}N_{\text{L}} \right] \frac{C_{\text{F}}\alpha_{\text{s}}^2(\mu)}{\pi^2} + O(\alpha_{\text{s}}^3). \quad (4.15)$$

This same result has been obtained by the authors of [50,51], in a calculation done entirely within the EFT. Given that we have made the subtractions, and now have renormalised expressions, we will deal with $\omega = 0$ below.

If we now write eqn (4.14) as

$$\frac{d \ln(Z_2^{\text{MS}}/\tilde{Z}_2^{\text{MS}})}{d \ln \mu} = 2 \sum_{n=1}^{\infty} e_n \tilde{a}^n \quad \begin{cases} e_1 = 4 \\ e_2 = 82 - \frac{44}{9} N_L \end{cases}$$

and the beta function as

$$\frac{d \ln \alpha_s}{d \ln \mu} = -2 \sum_{n=1}^{\infty} b_n \tilde{a}^n \quad \begin{cases} b_1 = 11 - \frac{2}{3} N_L \\ b_2 = 102 - \frac{38}{3} N_L \end{cases},$$

we can divide the two to obtain

$$\begin{aligned} \frac{\partial \ln r}{\partial \ln \tilde{a}} &= -\frac{e_1 \tilde{a} + e_2 \tilde{a}^2}{b_1 \tilde{a} + b_2 \tilde{a}^2} \\ &= -\frac{e_1}{b_1} - \tilde{a} \left(\frac{e_2}{b_1} - \frac{e_1 b_2}{b_1^2} \right). \end{aligned}$$

Integrating this, and exponentiating,

$$r(\mu) \propto \frac{1}{\tilde{a}(\mu)^{e_1/b_1}} \left(1 + \frac{e_2 b_1 - e_1 b_2}{b_1^2} \tilde{a}(\mu) + O(\tilde{a}^2) \right),$$

which gives

$$R(\mu) = \frac{r(\mu)}{Z_2^L} = R(M) \left(\frac{\alpha_s(M)}{\alpha_s(\mu)} \right)^{e_1/b_1} \frac{1 + E_2 \alpha_s(\mu)/\pi + O(\alpha_s^2(\mu))}{1 + E_2 \alpha_s(M)/\pi + O(\alpha_s^2(M))}$$

with

$$\begin{aligned} E_2 &= \frac{e_2 b_1 - e_1 b_2}{4 b_1^2} \\ &= \frac{175}{162} \text{ or } \frac{4253}{3750} \text{ for } N_L = 3 \text{ or } 4. \end{aligned}$$

The constant $R(M)$ is not arbitrary, and is fixed by the finite parts of the renormalised expressions for $r(\mu)$ and Z_2^L . Using these, we find $R(M)$ to be

$$R(M) = 1 + \frac{4}{3} \bar{\alpha}_s(M)/\pi + K_2 (\bar{\alpha}_s(M)/\pi)^2 + O(\bar{\alpha}^3), \quad (4.16)$$

where

$$\begin{aligned} K_2 &= \frac{2}{9} \pi^2 \ln 2 - \frac{1}{3} \zeta(3) + \frac{7}{6} \zeta(2) + \frac{4663}{288} - \left(\frac{1}{3} \zeta(2) + \frac{113}{144} \right) N_L \\ &\approx 19.23 - 1.33 N_L. \end{aligned}$$

4.5 Intermediate mass fermions

As in section 3.5, we must consider the effect on the renormalisation coefficient Z_2 of intermediate mass quarks in the loop in diagram 4d. We find dilogarithms again, but because of IR singularities in Z_2 , they come from a finite integral, expressing the fermion contribution to the zero-momentum gauge boson propagator. We find that the contribution to Z_2 of a single fermion of mass $M_i = rM$ is

$$\begin{aligned} \Delta Z_2 = & \left(\frac{\alpha_0}{\pi M^{2\omega}} \right) 2T_F C_F \left(\frac{1}{8\omega^2} + \frac{19 - 24 \ln r}{96\omega} \right. \\ & \left. + \frac{1}{4} \ln^2 r - \frac{11}{24} \ln r + \frac{1}{8} \zeta(2) + \frac{59}{192} + \bar{\Delta}(r) + O(\omega) \right) \end{aligned} \quad (4.17)$$

in terms of the bare coupling. The results [47] are similar to eqn (3.35),

$$\begin{aligned} \bar{\Delta}(r) &= \frac{1}{8} \int_0^1 dy \frac{(2+y)(1-y)}{y} \bar{\Pi}(r^2(1-y)/y^2) \\ \bar{\Pi}(z) &= 2(1-2z)\sqrt{1+4z} \operatorname{arccoth} \sqrt{1+4z} + 4z - \frac{5}{3} \end{aligned} \quad (4.18)$$

which was evaluated by Broadhurst to give [47]

$$\begin{aligned} \bar{\Delta}(r) &= \frac{1}{8}(r+1)(6r^3 - r^2 + r + 2)L_+(r) + \frac{1}{8}(r-1)(6r^3 - r^2 + r - 2)L_-(r) \\ &\quad + \frac{19}{24} \ln r + \frac{229}{288} + \left(\frac{1}{2} \ln r + \frac{7}{8} \right) r^2 \\ &= \sum_{n=1}^{\infty} (-2G(n) \ln r + G'(n)) r^{-2n}, \quad r \geq 1 \end{aligned}$$

with $G(n) = 3(n^2 - 1)/4n(n+2)(2n+1)(2n+3)$, which we have checked numerically. Setting $r = 1$ and substituting $\bar{\Delta}(1) = \frac{481}{288} - \zeta(2)$ in eqn (4.17) reproduces the expression (not given here) for the unrenormalised coefficient of $T_F N_F$ in Z_2 , which improves our confidence in both results.

The limiting behaviour of $\bar{\Delta}(r)$ is

$$\begin{aligned} \bar{\Delta}(r) &= \frac{1}{30} r^{-2} + O(r^{-4} \ln r) \\ \bar{\Delta}(r) &= \frac{1}{4} \ln^2 r + \frac{19}{24} \ln r + \frac{1}{4} \zeta(2) + \frac{229}{288} + O(r). \end{aligned} \quad (4.19)$$

Note that this is (infrared) divergent as $r \rightarrow 0$, unlike the corresponding term in eqn (3.37). This means that we cannot derive the contributions of massless fermions from this intermediate mass calculation. This complication does not affect the derivation of the gauge invariance of Z_2 , as the fermion loop in the gauge boson propagator in diagram 4d is gauge invariant, however the contribution is calculated.

4.6 Summary

We further extended the method of chapter 3 to cope with the integrals which appear when we turn our attention to wavefunction renormalisation. After calculating the renormalisation constant Z_2 , we found that it is gauge invariant to the two loop order, and relatively simply related to the pole-mass renormalisation constant Z_M . This led us to speculate that there is some hidden principle at work, which will be found to guarantee gauge invariance of Z_2 to all orders.

We briefly connected our results to recent work on EFT.

Chapter 5

Summary

In this thesis, we have described how we have extended the method of integration by parts to massive integrals, and used it to calculate the ratio of the running mass to the pole mass, and the on-shell wavefunction renormalisation constant, for fermions in both QED and QCD. The first is (eqns (3.7) and (3.31))

$$\begin{aligned} \frac{M}{m(M)} &= 1 + \frac{4}{3}d \frac{\alpha_s(M)}{\pi d} \\ &+ \left[\left(\frac{1}{9}\pi^2 \ln 2 - \frac{19}{36}\pi^2 - \frac{1}{6}\zeta(3) + \frac{665}{144} \right) d^2 + \left(\frac{1}{3}\pi^2 + \frac{71}{24} \right) d \right] \left(\frac{\alpha_s(M)}{\pi d} \right)^2 \\ &+ O(\alpha_s^3), \end{aligned}$$

where $d = 12/(33 - 2N_F)$ is sensitive to the number of quark flavours. The numerical value of the ratio $M/m(1 \text{ GeV})$ for the strange quark can be large, showing that a small running mass $m(\mu)$ is consistent with a much larger pole mass or, equivalently, that a small current mass is consistent with a large constituent mass, leaving a proportion to be accounted for non-perturbatively which is smaller than previous estimates [35].

Also, we discovered that the wavefunction renormalisation constant, Z_2 , is gauge invariant to two loops, although no argument exists to explain why this should be so. This, as well as an unexpectedly simple relationship between Z_2 and the pole mass renormalisation constant Z_M , is the result of such complicated cancellations that we speculate that it should be possible to find such an explanation. As a consequence of the gauge invariance, we can find an expression for the difference between the MS-renormalised wavefunction anomalous dimensions γ_F and those in a static-quark effective field theory $\tilde{\gamma}_F$, and from this difference, calculate $\tilde{\gamma}_F$ to be (eqn (4.15))

$$\tilde{\gamma}_F = \frac{(a-3)C_F\alpha_s(\mu)}{2\pi} + \left[\left(\frac{a^2}{32} + \frac{a}{4} - \frac{179}{96} \right) C_A + \frac{2}{3}T_F N_L \right] \frac{C_F\alpha_s^2(\mu)}{\pi^2} + O(\alpha_s^3).$$

Appendix A

Three loop relation of quark $\overline{\text{MS}}$ and pole masses

The following is a facsimile of ref [43].

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Three-loop relation of quark $\overline{\text{MS}}$ and pole masses

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Abstract. We calculate, exactly, the next-to-leading correction to the relation between the $\overline{\text{MS}}$ quark mass, \bar{m} , and the scheme-independent pole mass, M , and obtain

$$\frac{M}{\bar{m}(M)} \approx 1 + \frac{4}{3} \frac{\bar{\alpha}_s(M)}{\pi} + \left[16.11 - 1.04 \sum_{i=1}^{N_F-1} (1 - M_i/M) \right] \left(\frac{\bar{\alpha}_s(M)}{\pi} \right)^2 + O(\bar{\alpha}_s^3(M)),$$

as an accurate approximation for $N_F - 1$ light quarks of masses $M_i < M$. Combining this new result with known three-loop results for $\overline{\text{MS}}$ coupling constant and mass renormalization, we relate the pole mass to the $\overline{\text{MS}}$ mass, $\bar{m}(\mu)$, renormalized at arbitrary μ . The dominant next-to-leading correction comes from the finite part of on-shell two-loop mass renormalization, evaluated using integration by parts and checked by gauge invariance and infrared finiteness. Numerical results are given for charm and bottom $\overline{\text{MS}}$ masses at $\mu = 1$ GeV. The next-to-leading corrections are comparable to the leading corrections.

1 Introduction

QCD is well on the way to providing us with a quantitative theory of strong interactions. It becomes therefore the more important to fix the free parameters of the theory, namely the coupling and quark masses, accurately from experiment. In this paper we address ourselves to aspects of the problem of determining quark masses.

For very heavy quarks, the non-relativistic bound-state picture is expected to be valid. The relevant mass to be used, for example in the Balmer formula, is the pole mass. The pole mass, M , is a gauge-invariant, infrared-finite, renormalization-scheme-independent quantity [1]. It is a physically meaningful parameter, despite the confinement of colour, as long as the heavy quark is not exactly on shell. Typical values of these so-called consti-

tuents masses are [2, 3]

$$M_c \approx 1.46 \text{ GeV}, \quad M_b \approx 4.7 \text{ GeV} \quad (1)$$

for the charm and bottom quarks.

An alternative gauge-invariant mass is $\bar{m}(\mu)$, the mass of the modified minimal subtraction ($\overline{\text{MS}}$) scheme, renormalized at a scale μ . This so-called current mass is used as a parameter in kinematical situations with a large euclidean momentum $Q \sim \mu > M$, so as to absorb large logarithms that would otherwise render perturbation theory invalid. At the one-loop level, its scale dependence is given by [1]

$$\frac{d \log \bar{m}(\mu)}{d \log \mu} = -2\bar{\alpha}_s(\mu)/\pi + O(\bar{\alpha}_s^2(\mu))$$

where the renormalized $\overline{\text{MS}}$ coupling is given, to lowest order, by $\bar{\alpha}_s(\mu) \approx \pi d / \log(\mu^2/\Lambda^2)$, with $d \equiv 12/(33 - 2N_F)$ for N_F active quark flavours. Integrating the renormalization group equation for the $\overline{\text{MS}}$ mass, one encounters the scheme-dependent, renormalization-group-invariant mass \hat{m} as a constant of proportionality in [1]

$$\bar{m}(\mu) = \hat{m} \left(\frac{2\bar{\alpha}_s(\mu)}{\pi d} \right)^d (1 + O(\bar{\alpha}_s(\mu))).$$

To lowest order, the pole mass M is approximated by the $\overline{\text{MS}}$ mass $\bar{m}(M)$, renormalized at the pole mass, giving the one-loop relation [1]

$$\bar{m}(\mu) \approx M [\bar{\alpha}_s(\mu)/\bar{\alpha}_s(M)]^d \quad (2)$$

whose leading and next-to-leading corrections we calculate here. Leading corrections to (2) were taken into account in [3], where $\overline{\text{MS}}$ masses

$$\bar{m}_c(1 \text{ GeV}) \approx 1.42 \text{ GeV}, \quad \bar{m}_b(1 \text{ GeV}) \approx 6.3 \text{ GeV}$$

were found to correspond to pole masses (1) and to gauge-dependent 'euclidean' masses $m_c(p^2 = -M_c^2) \approx 1.26 \text{ GeV}$, $m_b(p^2 = -M_b^2) \approx 4.2 \text{ GeV}$, obtained from QCD sum rules [4, 5]. With $\Lambda = 0.18 \text{ GeV}$, the invariant masses were [3] $\hat{m}_c \approx 1.81 \text{ GeV}$, $\hat{m}_b \approx 7.9 \text{ GeV}$.

In this paper we calculate the next-to-leading corrections to the one-loop relationship (2), as follows. First,

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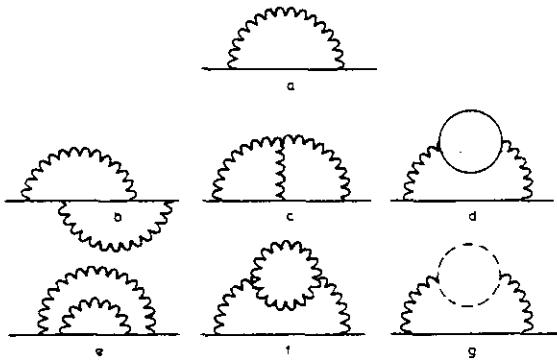


Fig. 1. Quark self-energy diagrams, to two loops

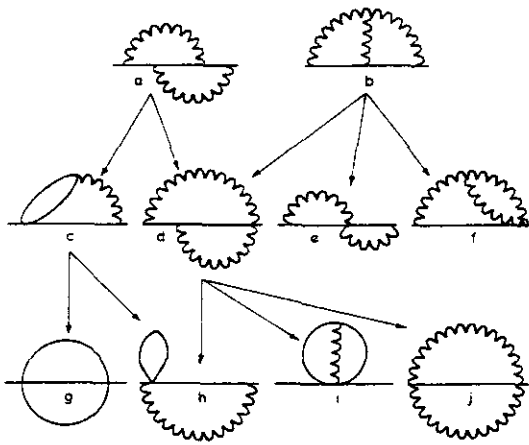


Fig. 2. Reduction of integrals (8, 9) by recurrence relations (10, 11)

in Sect. 2, we show how these corrections can be found from known three-loop results for $\overline{\text{MS}}$ coupling constant [6] and mass [7, 8] renormalization, together with the commensurate, but unknown, two-loop term in the relation

$$M/\bar{m}(M) = 1 + \frac{4}{3}\bar{\alpha}_s(M)/\pi + K\bar{\alpha}_s^2(M)/\pi^2 + O(\bar{\alpha}_s^3(M)). \quad (3)$$

In Sect. 3, we show how K is related to on-shell massive integrals derived from the diagrams of Fig. 1. Next, in Sect. 4, we extend the method of integration by parts [9] to dimensionally regularized, on-shell, two-loop, massive integrals [10]. Computer algebra then suffices to reduce all relevant two-loop integrals to gamma functions and a single difficult integral, evaluated in [11].

In Sect. 5, we use the techniques of Sects. 3 and 4 to obtain a gauge-invariant, infrared-finite, analytic result for K , by a route illustrated in Fig. 2. For simplicity we restrict the analysis to the situation of $N_F - 1$ massless quarks and a heavy quark of pole mass M running round the fermion loop of Fig. 1d. This restriction is removed in Sect. 6, where we obtain, in closed dilogarithmic form, the small further corrections due to finite lighter-quark masses. A numerical approximation to K is also given, accurate to 0.2%.

In Sect. 7, we present numerical results for charm and bottom quarks. The possibility of applying the same techniques to the relation between constituent and current

masses of the strange quark will be considered in a separate paper, since it requires attention to the non-perturbative contributions of strange-quark condensates [12, 13].

2 Sources of radiative correction

For the next-to-leading corrections to (2), we need the first three terms in three separate perturbative expansions.

First, to determine the $\overline{\text{MS}}$ coupling at the pole mass from its value estimated from experiments analyzed at some other scale μ , we use

$$\log \left[\frac{\mu^2}{M^2} \right] = \int_{\bar{\alpha}_s(\mu)/nd}^{\bar{\alpha}_s(M)/nd} \frac{dx}{b(x)} \quad (4)$$

where the first three terms of the beta function

$$b(x) = x^2 + \sum_{n=1}^{\infty} b_n x^{n+2}$$

are known from three-loop $\overline{\text{MS}}$ coupling constant renormalization [6], which gives

$$b_1 = -\frac{107}{16}d^2 + \frac{19}{4}d,$$

$$b_2 = -\frac{37117}{1536}d^3 + \frac{243}{32}d^2 + \frac{325}{96}d.$$

Next, to relate the $\overline{\text{MS}}$ masses at the scales μ and M , we use

$$\frac{\bar{m}(\mu)}{\bar{m}(M)} = \frac{c(\bar{\alpha}_s(\mu)/\pi d)}{c(\bar{\alpha}_s(M)/\pi d)} \quad (5)$$

where the first three terms of the anomalous mass dimension function

$$c(x) = x^d + \sum_{n=1}^{\infty} c_n x^{n+d}$$

are obtained from three-loop $\overline{\text{MS}}$ mass renormalization [7, 8], which gives

$$c_1 = \frac{23}{12}d^2 + (\frac{5}{6} - b_1)d,$$

$$c_2 = (\frac{55}{8}\zeta(3) - \frac{2591}{576})d^3 - (\frac{5}{2}\zeta(3) - \frac{313}{48})d^2 - \frac{1}{2}(\frac{35}{8} + b_2)d + \frac{1}{2}c_1(c_1 - b_1).$$

Finally, we need the first three terms in the expansion

$$M/\bar{m}(M) = 1 + \sum_{n=1}^{\infty} d_n(\bar{\alpha}_s(M)/\pi d)^n \quad (6)$$

whose leading and next-to-leading corrections are given by

$$d_1 = \frac{4}{3}d$$

$$d_2 \equiv Kd^2$$

of which only the leading correction has been given by previous authors [2, 3].

It is significant that each of the known coefficients $\{b_1, b_2, c_1, c_2, d_1\}$ is positive for $N_F = 3, 4, 5$ and hence has the effect of reducing the estimate (2) for given values of μ , $\bar{\alpha}_s(\mu)$ and $M < \mu$. The corrections to (4) increase $\bar{\alpha}_s(M)$ and hence work in concert with the corrections to (5) and

(6) to decrease $\bar{m}(\mu)$ at scales $\mu > M$. It is therefore possible to envisage a situation in which these five effects could work together to reduce the current-to-constituent mass ratio by a substantial factor. But before investigating this possibility, the sixth correction, d_2 , must be calculated.

Comparing (5) and (6), one sees that the finite part of two-loop on-shell mass renormalization, parametrized by d_2 , is commensurate with three-loop minimal subtraction, parametrized by c_2 . This reflects the fact that ultraviolet three-loop divergences can be obtained from massless two-loop two-point functions [9]. Thus the unknown d_2 is harder to calculate than the known c_2 , because the integration-by-parts algorithm [9], which has proved so successful for massless three- [6–8], four- [14] and five-loop [15, 16] counterterms, has not been developed for massive integrals to anything like the same extent. To our knowledge, integration by parts was first used for massive integrals by Grafe in [10], with results that we here correct and extend.

We shall use exclusively algebraic methods in $D \equiv 4 - 2\omega$ dimensions to express all the relevant on-shell two-loop diagrams in terms of gamma functions and a single massive Feynman integral whose value is required only for $D = 4$ and was found by Broadhurst in [11].

3 Reduction to on-shell integrals

In pursuit of the value of d_2 , we must evaluate the integrals corresponding to the diagrams of Fig. 1. For the two-loop diagrams, this task is at once analytically difficult and algebraically intensive. We show how the analytic burden may be reduced at the expense of a greater volume of algebra which can, however, be performed by computer.

To find d_2 , we first obtain an expression for the pole mass M in terms of m_0 , a_0 and g_0 ; the bare mass, gauge

with

$$C_1 = A_1(1),$$

$$C_2 = A_2(1) + A_1(1)[B_1(1) - 2A'_1(1)].$$

From the one-loop diagram 1a of Fig. 1, we find

$$A_1(1) = C_F \left(\frac{D-1}{D-3} \right) \Gamma(\omega)$$

$$B_1(1) = -C_F \left(\frac{a_0}{D-3} \right) \Gamma(\omega)$$

$$A'_1(1) = \frac{1}{2} C_F \left(D-1 - \frac{a_0}{D-3} \right) \Gamma(\omega)$$

where $C_F = (N_C^2 - 1)/2N_C$, for a gauge group $SU(N_C)$. Note that the gauge dependences of $B_1(1)$ and $A'_1(1)$ cancel in C_2 . The calculation of the gauge-invariant term $A_2(1)$ involves the two-loop diagrams 1b–1g of Fig. 1, and requires the techniques of the next section.

The integrals over loop momenta, involved in the calculation of the diagrams of Fig. 1, are such that the numerators of the integrals may be expressed as polynomials in the same Lorentz scalars as appear in the denominators, allowing cancellations and consequent simplification of the integrands. Thus we are left with a large number of primitive scalar integrals, which we evaluate on the bare mass-shell, at $m_0^2/p^2 = 1$.

4 Integration by parts

We now show how to extend to massive integrals the method of integration by parts of [9].

All of the two-loop integrals generated by the procedure of the previous section are of the form

$$\iint \frac{d^D k_1 d^D k_2}{k_1^{2\alpha_1} k_2^{2\alpha_2} (k_1^2 + 2p \cdot k_1)^{\alpha_3} (k_2^2 + 2p \cdot k_2)^{\alpha_4} [(k_1 + k_2)^2 + 2p \cdot (k_1 + k_2)]^{\alpha_5}} \equiv \pi^D (p^2)^{D-\sum \alpha_i} N(\alpha_1, \dots, \alpha_5), \quad (8)$$

parameter and coupling constant of the unrenormalized theory. The pole mass is defined by the condition that the unrenormalized Feynman propagator

$$S_F(p) \equiv \frac{i}{\not{p} - m_0 - \Sigma(p)}$$

has a pole as $p^2 \rightarrow M^2$. The term $\Sigma(p)$ is the proper self-energy, obtained, to two loops, by summing the diagrams of Fig. 1. We choose to expand it as follows:

$$\Sigma(p) = \sum_{n=1}^{\infty} \left[\frac{g_0^2}{(4\pi)^{D/2} p^{2\omega}} \right]^n \cdot [m_0 A_n(m_0^2/p^2) + (p - m_0) B_n(m_0^2/p^2)].$$

From the position of the pole in $S_F(p)$, we obtain an expansion

$$M = m_0 \left(1 + \sum_{n=1}^{\infty} \left[\frac{g_0^2}{(4\pi)^{D/2} M^{2\omega}} \right]^n C_n \right) \quad (7)$$

or

$$\iint \frac{d^D k_1 d^D k_2}{k_1^{2\alpha_1} (k_1 - k_2)^{2\alpha_2} k_2^{2\alpha_3} (k_1^2 + 2p \cdot k_1)^{\alpha_4} (k_2^2 + 2p \cdot k_2)^{\alpha_5}} \equiv \pi^D (p^2)^{D-\sum \alpha_i} M(\alpha_1, \dots, \alpha_5). \quad (9)$$

The treatment of integrals with such denominator structures is depicted in Fig. 2. In order to evaluate these integrals, we use recurrence relations to reduce them to sums of simpler integrals and a single irreducibility hard one.

The method we use is that of integration by parts. The key identity is

$$\iint d^D k_1 d^D k_2 \frac{\partial}{\partial k^\mu} [q^\mu f(k_1, k_2, p)] = 0$$

where $k \in \{k_1, k_2\}$, $q \in \{k_1, k_2, p\}$ and f is any scalar function of the minkowski loop momenta $k_{1,2}$ and the external

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momentum p . This identity generates six recurrence relations for a general two-loop integral. Two more independent relations can be obtained by differentiation with respect to p . Of the eight relations for each of (8) and (9), we find that the most useful are

$$(2\alpha_2 + \alpha_4 + \alpha_5 - D + \alpha_4 4^+ 2^- + \alpha_5 5^+ [2^- - 3^-]) \cdot N(\alpha_1, \dots, \alpha_5) = 0 \quad (10)$$

$$(2\alpha_2 + \alpha_1 + \alpha_4 - D + \alpha_1 1^+ [2^- - 3^-] + \alpha_4 4^+ [2^- - 5^-]) \cdot M(\alpha_1, \dots, \alpha_5) = 0 \quad (11)$$

where $1^\pm N(\alpha_1, \dots, \alpha_5) \equiv N(\alpha_1 \pm 1, \alpha_2, \dots, \alpha_5)$, etc.

In Fig. 2 we illustrate the application of these relations to diagrams 2a and 2b, which represent the general structures of (8) and (9). The figure is generated by applying (10) to diagrams 2a and 2c, and applying (11) to diagrams 2b and 2d. Diagrams 2e, 2f, 2h and 2j are easily evaluated as products of one-loop diagrams. For the less trivial bubble diagram 2i, we obtain a result of comparable form:

$$M(0, \alpha, 0, \beta, \gamma) = (-1)^{\alpha+\beta+\gamma+1} \frac{\Gamma\left(\alpha + \beta - \frac{D}{2}\right) \Gamma\left(\alpha + \gamma - \frac{D}{2}\right) \Gamma\left(\frac{D}{2} - \alpha\right) \Gamma(\alpha + \beta + \gamma - D)}{\Gamma(\beta) \Gamma(\gamma) \Gamma\left(\frac{D}{2}\right) \Gamma(2\alpha + \beta + \gamma - D)}$$

Diagram 2g represents integrals of the form (8), with $\alpha_{1,2} \leq 0$, which have, until now, resisted evaluation. In particular, we must consider the two cases $N(0, 0, \alpha_3, \alpha_4, \alpha_5)$ and $N(-1, 0, \alpha_3, \alpha_4, \alpha_5)$. By a systematic investigation of the eight independent recurrence relations, we obtain

$$\begin{aligned} x_3 N(-1, 0, \alpha_3 + 1, \alpha_4, \alpha_5) \\ = [2D - \alpha_3 - 2\alpha_4 - 2\alpha_5 - 2\alpha_4 4^- - 2\alpha_5 5^+] \cdot N(0, 0, \alpha_3, \alpha_4, \alpha_5), \end{aligned} \quad (12)$$

$$\begin{aligned} (\alpha_3 + \alpha_4 + \alpha_5 - 3D/2) N(-1, 0, \alpha_3, \alpha_4, \alpha_5) \\ = [2\alpha_3 + \alpha_4 + \alpha_5 - 2D + \alpha_4 4^+ [3^- - 5^-] \\ + \alpha_5 5^+ [3^- - 4^-] + (\alpha_3 + \alpha_4 + \alpha_5 - 3D/2) 3^-] \cdot N(0, 0, \alpha_3, \alpha_4, \alpha_5). \end{aligned} \quad (13)$$

By using these identities, we can reduce all the relevant integrals with the denominator structure of diagram 2g to products of one-loop integrals and a single two-loop integral, which we choose to express in terms of the finite integral

$$I(\omega) \equiv N(1, 1, 1, 1, 1).$$

The value of $I(\omega)$ is needed only at $\omega = 0$. This was determined by Broadhurst in [11], by analytically intensive methods, as

$$I(0) = \pi^2 \log 2 - \frac{3}{2} \zeta(3). \quad (14)$$

The techniques outlined in this section are sufficient to calculate all the diagrams of Fig. 1.

5 Calculation of d_2 with one massive quark

To evaluate the coefficient d_2 in expansion (6), we first calculate the second-order term $A_2(1)$, contributing to C_2

in expansion (7), and then relate m_0 to the renormalized mass \bar{m} .

To find $A_2(1)$, we express the two-loop diagrams of Fig. 1 as on-shell integrals in an arbitrary gauge and then use (10) to (13) to reduce these integrals to a single truly hard one, plus products of one-loop integrals, as described in Sect. 4 and illustrated in Fig. 2.

Using REDUCE 3.3 [17], running on a VAXCluster, we obtained

$$A_2(1) = \sum_{i=1}^4 \sum_{j=1}^3 N_i C_{ij} R_j \quad (15)$$

where

$$N_1 = C_A C_F, \quad N_2 = C_F^2, \quad N_3 = C_F N_F, \quad N_4 = C_F,$$

$$R_1 = \Gamma^2(\omega), \quad R_2 = \frac{\omega \Gamma^2(-\omega) \Gamma(-4\omega) \Gamma(2\omega) \Gamma(\omega)}{\Gamma(-2\omega) \Gamma(-3\omega)},$$

$$R_3 = I(\omega),$$

with $C_A = N_C$, for a gauge group $SU(N_C)$, and coefficients C_{ij} given in Table 1.

The structure R_1 is associated with diagrams 2e, 2h and 2i; R_2 with diagrams 2f and 2j; and R_3 with diagram

Table 1. Coefficients C_{ij} in (15)

$C_{11} =$	$\frac{-3(5D^3 - 58D^2 + 180D - 152)}{2(3D - 8)(3D - 10)(D - 3)}$
$C_{12} =$	$\frac{-4(4D^3 - 41D^2 + 122D - 104)}{3(3D - 8)(3D - 10)(D - 3)}$
$C_{13} =$	$\frac{4(D^2 - 7D + 8)(D - 3)(D - 6)}{(3D - 8)(3D - 10)}$
$C_{21} =$	$\frac{9D^5 - 84D^4 + 248D^3 - 175D^2 - 226D + 168}{(3D - 8)(3D - 10)(D - 3)^2}$
$C_{22} =$	$\frac{8(6D^4 - 78D^3 + 355D^2 - 677D + 454)}{3(3D - 8)(3D - 10)(D - 3)^2}$
$C_{23} =$	$\frac{-8(D^2 - 7D + 8)(D - 3)(D - 6)}{(3D - 8)(3D - 10)}$
$C_{31} =$	0
$C_{32} =$	$\frac{16(D - 2)}{3(3D - 8)(3D - 10)}$
$C_{33} =$	0
$C_{41} =$	$\frac{12(D^3 - 12D^2 + 50D - 68)}{(3D - 8)(3D - 10)(D - 3)(D - 6)}$
$C_{42} =$	$\frac{-32(D^3 - 9D^2 + 21D - 10)}{3(3D - 8)(3D - 10)(D - 3)(D - 6)}$
$C_{43} =$	$\frac{8(D^3 - 7D^2 + 6D + 16)(D - 4)}{(3D - 8)(3D - 10)(D - 6)}$

Table 2. Coefficients of leading and next-to-leading corrections

N_F	d_1	c_1	b_1	d_2	c_2	b_2
3	0.593	0.398	0.790	2.771	0.535	0.883
4	0.640	0.487	0.739	2.992	0.763	0.702
5	0.696	0.613	0.658	3.251	1.120	0.401

2g. The colour factor N_4 is due to a single massive quark in diagram 1d, whilst N_3 results from $N_F - 1$ massless quarks in the same diagram. Note that our result for $A_2(1)$ is gauge-invariant, in all dimensions D . The consequent gauge invariance of the pole mass of (7) provides a strong check on our procedures.

We can now find an expression for the coefficient d_2 of expansion (6), as follows. The bare coupling constant and mass are related to the normalized $\overline{\text{MS}}$ coupling and mass by

$$\begin{aligned} \frac{g_0^2}{4\pi} &= \left(\frac{\mu^2 e^\gamma}{4\pi} \right)^\omega \bar{\alpha}_s(\mu) \left[1 - \frac{\bar{\alpha}_s(\mu)}{\pi} \frac{1}{\omega} \right. \\ &\quad \cdot \left. \left[\frac{11}{12} C_A - \frac{1}{6} N_F \right] + O(\bar{\alpha}_s^2(\mu)) \right], \\ m_0 &= \bar{m}(\mu) \left[1 + \frac{\bar{\alpha}_s(\mu)}{\pi} \frac{1}{\omega} Z_{11} + \left(\frac{\bar{\alpha}_s(\mu)}{\pi} \right)^2 \right. \\ &\quad \cdot \left. \left\{ \frac{1}{\omega^2} Z_{22} + \frac{1}{\omega} Z_{21} \right\} + O(\bar{\alpha}_s^3(\mu)) \right]. \end{aligned}$$

The ratio of the pole mass (7) to the $\overline{\text{MS}}$ mass is a *finite* function of the *renormalized* coupling. Thus, from our on-shell result (15), we obtain the ultraviolet minimal subtractions

$$\begin{aligned} Z_{11} &= -\frac{3}{4} C_F \\ Z_{21} &= -\frac{97}{192} C_A C_F - \frac{3}{64} C_F^2 + \frac{5}{96} C_F N_F \\ Z_{22} &= \frac{11}{32} C_A C_F + \frac{9}{32} C_F^2 - \frac{1}{16} C_F N_F \end{aligned}$$

which agree with the results of the much simpler deep-euclidean calculation of Tarrach [1], confirming that (7) is free of infrared singularities.

Gauge invariance and infrared finiteness thus provide strong checks on our result

$$\begin{aligned} d_2 &= \left(\frac{1}{3} \pi^2 \log 2 - \frac{19}{36} \pi^2 - \frac{1}{6} \zeta(3) + \frac{665}{144} \right) d^2 + \left(\frac{1}{3} \pi^2 + \frac{7}{24} \right) d \\ &\approx -0.031 d^2 + 6.248 d \end{aligned} \quad (16)$$

obtained from the finite $\omega \rightarrow 0$ limit of (7), with the help of (14). In Table 2 we give the values of the expansion coefficients for $N_F = 3, 4, 5$. Note that d_2 dominates the next-to-leading corrections.

6 Lighter-quark mass corrections

In obtaining (16), we made the approximation that the masses of the $N_F - 1$ quarks with masses $M_i < M$ could be neglected in comparison with the heavy quark mass M . Here we calculate the corrections, due to finite lighter-quark masses, to the coefficient K in (3), which

has the form

$$K = K_0 + \sum_{i=1}^{N_F-1} \Delta(M_i/M) \quad (17)$$

where

$$\begin{aligned} K_0 &= \frac{1}{9} \pi^2 \log 2 + \frac{7}{18} \pi^2 - \frac{1}{6} \zeta(3) + \frac{3673}{288} - \left(\frac{1}{18} \pi^2 + \frac{71}{144} \right) N_F \\ &\approx 17.15 - 1.04 N_F \end{aligned} \quad (18)$$

is obtained from (16).

The finite-mass correction, $\Delta(M_i/M)$, can be obtained from $\Pi(M_i^2/Q^2)$, the finite gauge-invariant *difference* between the contributions of massive and massless quark loops to the one-loop gluon propagator at euclidean momentum Q . We find that

$$\begin{aligned} \Pi(z) &= 2(1-2z)\sqrt{1+4z} \operatorname{arccoth} \sqrt{1+4z} + \log z + 4z, \\ \Delta(r) &= \frac{1}{24} \int_0^1 dy \left(\frac{4-y^2}{1-y} \right) \Pi(r^2(1-y)/y^2). \end{aligned}$$

It is possible, though not easy, to reduce this integral to the elementary dilogarithms

$$\begin{aligned} L_{\pm}(r) &\equiv \int_0^1 dx \left(\frac{\log x - \log r}{x \pm r} \right) = \frac{1}{2} \log^2 r \\ &\quad + \sum_{n=1}^{\infty} \frac{(\mp 1)^n}{n^2} (r^n (n \log r - 1) + 2) \end{aligned}$$

in terms of which, we obtain, after much computer algebra, the dilogarithmic series

$$\begin{aligned} \Delta(r) &= \frac{1}{4} [\log^2 r + \frac{1}{3} \pi^2 - (\log r + \frac{3}{2}) r^2 \\ &\quad - (1+r)(1+r^3)L_+(r) - (1-r)(1-r^3)L_-(r)] \\ &= \frac{1}{3} \pi^2 r - \frac{1}{3} r^2 + \frac{1}{3} \pi^2 r^3 - \left(\frac{1}{4} \log^2 r - \frac{1}{24} \log r + \frac{1}{24} \pi^2 + \frac{151}{88} \right) r^4 \\ &\quad - \sum_{n=3}^{\infty} (2F(n) \log r + F'(n)) r^{2n} \end{aligned} \quad (19)$$

where $F(n) = 3(n-1)/4n(n-2)(2n-1)(2n-3)$. To a good approximation, $\Delta(r)/r$ can be treated as constant, since it varies little from $\pi^2/8 \approx 1.2$, at $r=0$, to $(\pi^2-3)/8 \approx 0.9$, at $r=1$. For the largest mass ratios encountered, namely $r \sim M_s/M_c \sim M_c/M_b \approx 0.3$, an intermediate value $\Delta(r)/r \approx 1.04$ is both accurate and convenient, allowing us to approximate the exact results (17) to (19) by

$$K \approx 16.11 - 1.04 \sum_{i=1}^{N_F-1} (1 - M_i/M) \quad (20)$$

which is accurate to 0.2%, giving $K_c \approx 13.3$ and $K_b \approx 12.4$, for charm and bottom quarks.

7 Results and conclusions

In Table 3 we give values of $\bar{m}(1 \text{ GeV})$, for various values of the pole mass M and the coupling $\bar{\alpha}_s(1 \text{ GeV})$, taking into account the next-to-leading ($L=3$) and leading ($L=2$) corrections to the lowest-order ($L=1$) relationship (2).

The method of calculation was as follows. The value of $\bar{\alpha}_s(M)$ was obtained from $\bar{\alpha}_s(1 \text{ GeV})$ by exact integration of (4), with L terms retained in the beta function $b(x)$.

$\bar{\alpha}_s$	0.25			0.30			0.35		
L	1	2	3	1	2	3	1	2	3
M									
4.80	5.89	5.74	5.65	6.08	5.97	5.87	6.27	6.21	6.12
4.70	5.75	5.61	5.51	5.94	5.83	5.73	6.12	6.06	5.97
4.60	5.62	5.47	5.38	5.80	5.68	5.59	5.98	5.91	5.82
1.50	1.59	1.48	1.41	1.61	1.49	1.39	1.63	1.50	1.38
1.45	1.53	1.42	1.35	1.55	1.43	1.33	1.57	1.43	1.32
1.40	1.47	1.36	1.29	1.49	1.37	1.27	1.50	1.37	1.25
0.55	0.49	0.42	0.36	0.48	0.38	0.30	0.47	0.34	0.23
0.50	0.44	0.36	0.31	0.43	0.33	0.24	0.41	0.29	0.17
0.45	0.39	0.31	0.26	0.37	0.27	0.19	0.36	0.23	0.11

Table 3. $\bar{m}(1 \text{ GeV})$ obtained from M , at L loops, with $\bar{\alpha}_s(1 \text{ GeV}) = 0.30 \pm 0.05$

The value of $\bar{m}(1 \text{ GeV})$ was obtained, as a multiple of $\bar{m}(M)$, by using (5), with L terms retained in the anomalous dimension function $c(x)$. The value of $\bar{m}(M)$ was obtained from M by using (3), with L terms retained, the third of which is given, to high accuracy, by our new result (20). Quarks of mass *greater* than M were ignored in Fig. 1d, since they decouple [18] from physical amplitudes at momenta of order M . Thus our only significant approximation is the neglect of higher-order terms, with coefficients $\{b_n, c_n, d_n | n \geq L\}$, in (4) to (6).

By this systematic procedure we avoid all reference to the $\overline{\text{MS}}$ renormalization-group invariants Λ and \bar{m} , whose extracted values are notoriously dependent on N_F and L . For $M = (4.7 \pm 0.1) \text{ GeV}$ we took $N_F = 5$, corresponding to $M \equiv M_b$. For $M = (1.45 \pm 0.05) \text{ GeV}$ we took $N_F = 4$, corresponding to $M \equiv M_c$. For the final three rows of Table 3, with $M = (0.5 \pm 0.05) \text{ GeV}$, we set $N_F = 3$, in order to indicate how much smaller than a constituent strange quark mass, of size $M_s \sim M_\phi/2 \approx M_K$, the current strange quark mass $\bar{m}_s(1 \text{ GeV})$ might be in *perturbation* theory.

From Table 3 it is apparent that the next-to-leading corrections to the charm and bottom current-to-constituent mass ratios are comparable to the leading corrections, considered by Narison [3]. The scale at which the coupling must be renormalized, so as to give an $O(\bar{\alpha}_s^2(\mu_0))$ correction to $M/\bar{m}(M)$ that *vanishes*, is $\mu_0 \equiv M \exp(-d_2/2d_1) \approx 0.10M$. For the charm quark, this is of the same order as the $\overline{\text{MS}}$ scale Λ , at which perturbation theory breaks down. Thus a significant next-to-leading correction is *unavoidable*.

This situation is similar to, though not as extreme as, that found in [19] for the next-to-leading corrections to quark-condensate contributions in QCD sum rules for the ϕ meson, and there interpreted by the authors as negating the approach of [4]. In the language of [19], our radiative corrections define a scale $\Lambda_{\text{eff}} \sim 10\Lambda$, below which perturbation theory is suspect. (In [19] this scale was of order 30Λ .) One might, however, adopt the more pragmatic attitude that, with $\bar{\alpha}_s(1 \text{ GeV}) \approx 0.3$, the 6% next-to-leading corrections of Table 3 for the charm quark are 'acceptably' small, whilst the 8% leading corrections to (2) may be 'accidentally' small. Little more can be said, without knowing next-to-next-to-leading

corrections, now available in e^+e^- annihilation [14], but here, with massive integrals, prohibitively expensive of labour.

Any attempt to reconcile the small current mass, $\bar{m}_s(1 \text{ GeV}) \approx 0.2 \text{ GeV}$, with the larger constituent mass, $M_s \sim M_\phi/2 \approx M_K \approx 0.5 \text{ GeV}$, of the strange quark [2], must address itself to this perturbative question, as well as to estimates of the non-perturbative effects of [13], which *further* reduce the current-to-constituent mass ratio. We postpone a detailed consideration of these issues to a subsequent paper and here merely note that the question of how much of the difference between $\bar{m}_s(1 \text{ GeV}) \approx 0.2 \text{ GeV}$ and $M_s \sim 0.5 \text{ GeV}$ is perturbative, and how much non-perturbative in origin, is still open.

In conclusion, we have obtained the exact results (17) to (19), and the accurate approximation (20), for the coefficient, K , of $\bar{\alpha}_s^2(M)/\pi^2$ in the expansion (3) of the ratio of the pole mass, M , to the $\overline{\text{MS}}$ mass $\bar{m}(M)$. The effect of K , in reducing the current-to-constituent mass ratio $\bar{m}(\mu)/M$, is augmented by three-loop $\overline{\text{MS}}$ mass and coupling constant renormalizations at $\mu > M$, but opposed by them at $\mu < M$.

With $\bar{\alpha}_s(1 \text{ GeV}) \approx 0.3$, the next-to-leading corrections reduce $\bar{m}_c(1 \text{ GeV})/M_c$ and $\bar{m}_b(1 \text{ GeV})/M_b$ by 6% and 2%, respectively, and are comparable to the leading corrections of 8% and 2%, respectively. The applicability of perturbation theory, in obtaining $\bar{m}_c(1 \text{ GeV})/M_c \approx 0.9$, is open to question, as is the attribution [12] of the small value of $\bar{m}_s(1 \text{ GeV})/M_s \sim 0.4$ dominantly to non-perturbative strange-quark condensation.

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Appendix B

Gauge-invariant on-shell Z_2 in QED, QCD and the effective field theory of a static quark

The following is a facsimile of ref [47].

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Gauge-invariant on-shell Z_2 in QED, QCD and the effective field theory of a static quark

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Abstract. We calculate the *on-shell* fermion wave-function renormalization constant Z_2 of a general gauge theory, to two loops, in D dimensions and in an arbitrary covariant gauge, and find it to be gauge-invariant. In QED this is consistent with the dimensionally regularized version of the Johnson-Zumino relation: $d \log Z_2 / d a_0 = i(2\pi)^{-D} e_0^2 \int d^D k / k^4 = 0$. In QCD it is, we believe, a new result, strongly suggestive of the cancellation of the gauge-dependent parts of non-abelian UV and IR anomalous dimensions to all orders. At the two-loop level, we find that the anomalous dimension γ_F of the fermion field in minimally subtracted QCD, with N_L light-quark flavours, differs from the corresponding anomalous dimension $\tilde{\gamma}_F$ of the effective field theory of a static quark by the gauge-invariant amount

$$\gamma_F - \tilde{\gamma}_F \equiv \mu \frac{d}{d\mu} \log \left(\frac{Z_2^{\text{MS}}(\mu)}{\tilde{Z}_2^{\text{MS}}(\mu)} \right) \\ = 2 \frac{\tilde{\alpha}_s(\mu)}{\pi} + \left(\frac{41}{4} - \frac{11}{18} N_L \right) \frac{\tilde{\alpha}_s^2(\mu)}{\pi^2} + O(\tilde{\alpha}_s^3).$$

A complete description of two-loop on-shell renormalization of one-lepton QED, in D dimensions, is also given. More generally, we show that there is no need of integration in the two-loop calculation of on-shell two- and three-point functions.

1 Introduction

In a massive scalar field theory, the on-shell renormalization scheme is defined by identifying the wave-function renormalization constant with the constant Z in the LSZ [1] asymptotic relation of the bare Heisenberg field ϕ_0 to the in and out fields $\phi_{\text{in,out}}$ which create correctly normalized initial and final physical states. In the sense

of 'weak' convergence [2] one may write

$$\phi_0(x) \rightarrow \sqrt{Z} \phi_{\text{in,out}}(x) \quad \text{as } x_0 \rightarrow \mp \infty.$$

The on-shell renormalization $\phi_0 = \sqrt{Z} \phi$ then ensures that S-matrix elements are given by on-shell limits of truncated (i.e. proper) renormalized Green functions [3]. In any other scheme, such as a minimal subtraction (MS) scheme with wave-function renormalization constant $Z^{\text{MS}}(\mu)$, it is necessary to multiply a renormalized Green function by $(Z^{\text{MS}}(\mu)/Z)^{N_E/2}$ to obtain the corresponding S-matrix element for a process with N_E external particles. In massive scalar field theory, such a correction factor has a finite perturbative expansion in terms of the renormalized mass and coupling, which is most easily found from the residue $Z/Z^{\text{MS}}(\mu)$ of the renormalized propagator at $p^2 = M^2$, where M is the pole (i.e. physical) mass. This is because Z is the residue at the pole of the bare propagator [4]. Formally, one may regard Z as the probability for 'finding' the bare particle in the dressed one and use a dispersion relation [4] to show that $Z < 1$.

The situation in a gauge theory is rather different. If the ultraviolet (UV) infinities of the fermion propagator are removed by the MS renormalizations $\psi_0 = \sqrt{Z_2^{\text{MS}}(\mu)} \psi$ and $m_0 = Z_m^{\text{MS}}(\mu) \bar{m}(\mu)$ of the bare-fermion field and mass, the pole mass M has a finite perturbative expansion [5], but the residue at the pole does not, because of the 'infrared catastrophe' [6] of accumulating branch points of cuts with intermediate states consisting of one fermion and any number of gauge bosons. It is therefore straightforward to compute [7] the finite perturbative relation between the MS mass $\bar{m}(\mu)$ and the pole mass M , but much more problematic to give a meaningful expression for the factor $Z_2^{\text{MS}}(\mu)/Z_2$, required to convert Green functions of the MS scheme into S-matrix elements, since it contains infrared (IR) singularities. In QED, these are cancelled by the Bloch-Nordsieck [8] mechanism of incoherently adding probabilities for low-energy photon emission to the probability given by the square of the S-matrix element, thereby obtaining finite answers to experimentally meaningful questions

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[9]. In QCD, however, this mechanism fails for certain [10] initial states.

In a previous paper [7] we have investigated the relation between MS and on-shell mass renormalization, by combining the results of three-loop MS mass renormalization [11] with our new results for the finite part of on-shell two-loop mass renormalization. The latter are commensurable with the former and turn out to dominate them numerically.

In this paper we use the same technique of simultaneous dimensional regularization of UV and IR singularities to calculate on-shell two-loop fermion wave-function renormalization, in an arbitrary covariant gauge of an arbitrary gauge theory in an arbitrary dimension $D \equiv 4 - 2\omega$, and show that it is gauge-invariant. In QED there exists an argument [12] why this should be the case. In the case of a non-abelian theory such as QCD, we know of no such general argument, but are encouraged by our two-loop result to believe that dimensional regularization renders Z_2 gauge-invariant to all orders, thereby respecting its formal probabilistic interpretation. We hope that a proof of this may eventually be forthcoming from non-abelian functional integration.

The utility of our result is demonstrated by deriving from it the two-loop anomalous dimension $\tilde{\gamma}_F$ of the field of a static quark, interacting with gluons and massless quarks in the effective field theory (EFT) obtained in the limit $M \rightarrow \infty$ [13–15]. The gauge invariance of Z_2 implies that the corresponding anomalous dimension γ_F of conventional QCD differs from $\tilde{\gamma}_F$ by a gauge-invariant amount, which is simply calculable from the Laurent expansion of Z_2 . Confirmation of our result for $\tilde{\gamma}_F$ has recently been obtained by Broadhurst and Grozin [16], working exclusively within EFT.

The utility of our method is further demonstrated by obtaining, using only computer algebra, all the two-loop on-shell renormalization constants of one-lepton QED, in any dimension D , in terms of Γ functions and a single D -dimensional integral, $I(\omega)$, whose $D \rightarrow 4$ limit, $I(0) = \pi^2 \log 2 - \frac{3}{2}\zeta(3)$, was found by one of us [17]. More generally, on-shell two-loop three-point functions, such as that giving the $O(\alpha^2)$ corrections [18] to $g-2$, may be expressed in terms of Γ functions and $I(\omega)$, which may itself be expanded through $O(\omega^2)$ using exclusively algebraic methods [19].

The remainder of this paper is organized as follows.

In Sect. 2 we show how Z_2 and $Z_m \equiv m_0/M$ are reduced to integrals on the bare-mass shell, when all but one of the fermions are massless. The one-loop integrals are trivially evaluated. The two-loop integrals are related by recurrence relations to three general structures, of which only $I(\omega)$ is not reducible to Γ functions. Hence we obtain the Laurent expansions of Z_2 and Z_m as $\omega \rightarrow 0$, including important finite terms.

In Sect. 3 we evaluate the effects of non-trivial fermion mass ratios, since these are of importance in QED. Only when one has a finite mass ratio, such as M_e/M_μ , is it necessary to resort to Spence functions.

In Sect. 4 we derive the two-loop EFT anomalous dimension $\tilde{\gamma}_F$ from Z_2 and the known [20] two-loop QCD anomalous dimension γ_F .

In Sect. 5 we give all the two-loop on-shell renormalization constants of QED and indicate other QED calculations which are reduced to algebra by our method.

In Sect. 6 we summarize our findings and present conclusions.

2 Expansion in the bare coupling

We achieve the expansion, to $O(g_0^4)$ in the bare coupling, in four stages. First we determine which combinations of on-shell integrals enter the two-loop expansion of Z_2 and Z_m via the bare-fermion self energy $\Sigma(p)$ and its derivatives on the bare-mass shell, $p^2 = m_0^2$. Then we evaluate the one-loop terms and show that $Z_2 = Z_m + O(g_0^4)$. Next we evaluate the two-loop integrals in D dimensions, by computer algebra. Finally we give the Laurent expansions of Z_2 and Z_m as $D \rightarrow 4$.

Throughout this Sect. we assume that the fermion loop in the gauge boson propagator involves only the external fermion, of mass M , and (if desired) N_L massless fermions, so that we are evaluating integrals which depend only upon the dimension D and bare gauge parameter a_0 . Non-trivial fermion mass ratios will be treated in Sect. 3. Coupling constant renormalization will be treated in Sects. 4 and 5, for QCD and QED respectively.

2.1 Reduction to on-shell integrals

Starting from the perturbative expansion of the bare self energy, $\Sigma(p)$, in terms of the bare coupling constant, g_0 , the bare mass, m_0 , and the bare gauge parameter, a_0 , we calculate Z_2 by finding the residue, at the pole mass M , of the bare Feynman propagator

$$\begin{aligned} S_F(p) &\equiv \frac{1}{\not{p} - m_0 - \Sigma(p)} \\ &= \frac{Z_2}{\not{p} - M} + (\text{terms regular at } p^2 = M^2) \end{aligned} \quad (1)$$

in $D \equiv 4 - 2\omega$ dimensions. The essence of dimensional continuation is to regulate both ultraviolet and [21] infrared singularities by the introduction of a *single* dimensionless parameter, D , which formally preserves both the Lorentz invariance and the gauge invariance of the action, making no attempt to separate the resultant $\omega \rightarrow 0$ singularities into $1/\omega_{UV}$ and $1/\omega_{IR}$ terms. Whilst such a separation may be possible at the one-loop level, it is quite impractical at two loops, where the method of integration by parts [22] routinely introduces *extra* factors of $1/\omega$ in the process of reducing integrals to known forms. Computationally, the prescription is very well defined: one merely instructs a program like REDUCE [23] that $g_\mu^\mu = D$ and gives it a master formula, and/or a set of recurrence relations [7], sufficient to translate all possible terms encountered in momentum-space integrands, generated by the Feynman rules, into functions of D which correspond to the integrals.

We find it convenient to expand the bare self energy as

$$\Sigma(p) = \sum_{n=1}^{\infty} \left[\frac{g_0^2}{(4\pi)^{D/2} p^{2\omega}} \right]^n \cdot (m_0 A_n(m_0^2/p^2) + (\not{p} - m_0) B_n(m_0^2/p^2)) \quad (2)$$

where A_n and B_n are dimensionless functions of the dimension, D , the gauge parameter, a_0 , and the dimensionless variable m_0^2/p^2 . Note that the coupling constant has mass dimension ω , which has been cancelled by a power of the time-like momentum p , before taking the limit $p^2 \rightarrow m_0^2$. Then the coefficients of the expansions

$$\frac{m_0}{M} \equiv Z_m = 1 + \sum_{n=1}^{\infty} \left[\frac{g_0^2}{(4\pi)^{D/2} M^{2\omega}} \right]^n M_n \quad (3)$$

$$(\not{p} - M) S_F(p)|_{\not{p}=M} \equiv Z_2 = 1 + \sum_{n=1}^{\infty} \left[\frac{g_0^2}{(4\pi)^{D/2} M^{2\omega}} \right]^n F_n \quad (4)$$

are determined by combinations of A_n and B_n and their derivatives on the bare-mass shell. Specifically we find, by substitution of (2) in (1), that the following combinations are required at the two-loop level:

$$\begin{aligned} M_1 &= -A_1 \\ M_2 &= -A_2 + A_1(A_1 + 2A'_1 - B_1) \\ F_1 &= B_1 - 2\omega A_1 - 2A'_1 \\ F_2 &= B_2 - 4\omega A_2 - 2A'_2 + (2A'_1 - B_1)^2 + 4A_1(A'_1 - B'_1) \\ &\quad + 2(1 + 2\omega)A_1(\omega A_1 + 3A'_1) - 6\omega A_1 B_1 \end{aligned}$$

with all the A and B terms evaluated at $m_0^2/p^2 = 1$, for which the calculation of integrals is much simplified. To find the derivatives with respect to m_0^2/p^2 , one has merely to differentiate diagrams one or two times with respect to the bare mass, before going on shell, thereby merely making zero-momentum insertions in internal fermion propagators.

2.2 One-loop result

From the one-loop integrals of Fig. 1a one easily obtains

$$\begin{aligned} A_1 &= C_F \left(\frac{D-1}{D-3} \right) \Gamma(\omega) \\ A'_1 &= C_F \left(\frac{(D-1)(D-3) - a_0}{2(D-3)} \right) \Gamma(\omega) \\ A''_1 &= C_F \left(\frac{(D-6)((D-1)(D-4) - 2a_0)}{4(D-5)} \right) \Gamma(\omega) \\ B_1 &= -C_F \left(\frac{a_0}{D-3} \right) \Gamma(\omega) \\ B'_1 &= -C_F \left(\frac{(D-2)a_0}{2(D-3)} \right) \Gamma(\omega) \end{aligned}$$

where $C_F = (N_C^2 - 1)/2N_C$ for a gauge group $SU(N_C)$. Hence we obtain the one-loop coefficients

$$M_1 = F_1 = -C_F \left(\frac{D-1}{D-3} \right) \Gamma(\omega) \quad (5)$$

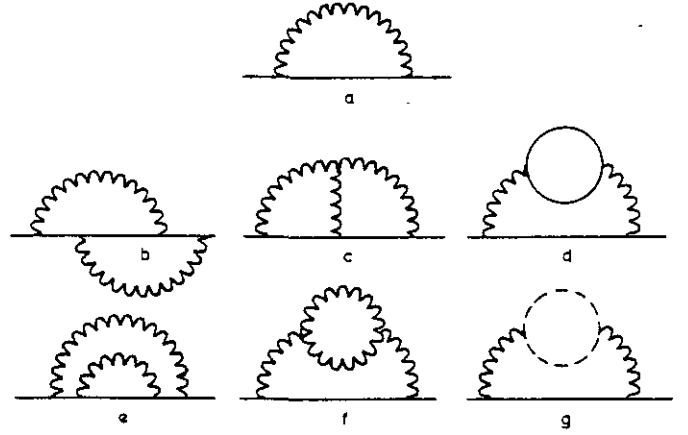


Fig. 1. Fermion self-energy diagrams, to two loops

showing that Z_m and Z_2 are gauge-invariant and equal at the one-loop level.

As there is no non-abelian coupling at this order, the one-loop gauge invariance of Z_2 may be obtained directly from the dimensionally regularized version of the QED Johnson-Zumino identity [12, 24]

$$d \log Z_2 / d a_0 = i(2\pi)^{-D} e_0^2 \int d^D k / k^4 = 0 \quad (6)$$

which derives from an earlier analysis by Landau and Khalatnikov [25] of the transformation of Green functions under covariant gauge transformations. Note that Z_2 is therefore gauge-invariant to all orders in QED. We are not aware of a nonabelian generalization of (6) that would ensure the gauge invariance of Z_2 to all orders in QCD.

Whilst the one-loop gauge invariance of Z_2 is to be expected from QED, we have no explanation of the remarkable coincidence

$$Z_2 = Z_m + O(g_0^4) \quad (7)$$

which means that, to leading order, the mass term $\bar{\psi}_0 m_0 \psi_0$, in the bare Lagrangian density, is renormalized by a factor $Z_2 Z_m$ which is the square of the factor Z_2 by which the kinetic energy term $\bar{\psi}_0 i \not{\partial} \psi_0$ is renormalized. We shall show that this 'virial' relationship does not persist at two loops, where it is replaced by a simple relation between the contributions to Z_2 and Z_m with three-fermion intermediate states.

There is a rather instructive consistency check on (7), provided by conventional MS renormalization. With \bar{a} and $\bar{\alpha}_s$ representing the gauge parameter and coupling renormalized at scale μ in the MS scheme, the anomalous dimensions [26]

$$\gamma_F(\bar{a}, \bar{\alpha}_s) \equiv \frac{d \log Z_F^{\text{MS}}(\mu)}{d \log \mu} = \frac{\bar{a} C_F \bar{\alpha}_s}{2\pi} + O(\bar{\alpha}_s^2) \quad (8)$$

$$\gamma_m(\bar{\alpha}_s) \equiv \frac{d \log Z_m^{\text{MS}}(\mu)}{d \log \mu} = \frac{3 C_F \bar{\alpha}_s}{2\pi} + O(\bar{\alpha}_s^2) \quad (9)$$

are indeed equal at the one-loop level in precisely that gauge for which there is no [9] infrared catastrophe, namely the Yennie gauge [27] with $\bar{a} = 3$.

It was shown by Abrikosov [6] that in QED the electron propagator has a one-loop *infrared* anomalous dimension $\tilde{\gamma}_F = (a-3)\alpha/2\pi$. Other authors [28] verified that this result is spin-independent. Recently it has become possible to give a precise definition [16] to $\tilde{\gamma}_F$ in EFT, in analogy with (8), namely

$$\tilde{\gamma}_F(\bar{a}, \bar{\alpha}_s) \equiv \frac{d \log \tilde{Z}_2^{MS}(\mu)}{d \log \mu} = \frac{(\bar{a}-3) C_F \bar{\alpha}_s}{2\pi} + O(\bar{\alpha}_s^2) \quad (10)$$

where $\tilde{Z}_2^{MS}(\mu)$ gives the minimal subtractions which regularize the fermion propagator in the effective field theory [13, 14] of a static fermion, obtained as $M \rightarrow \infty$. In EFT it is trivial [16] to obtain the one-loop Abrikosov result (10) by repeating the one-loop self-energy calculation of Eichten and Hill [14] in an arbitrary covariant gauge. The coincidence of the one-loop $1/\omega$ singularities in (7) may thus be written as

$$\gamma_F - \tilde{\gamma}_F = \gamma_m + O(\bar{\alpha}_s^2). \quad (11)$$

The relation of $\gamma_F - \tilde{\gamma}_F$ to the gauge-invariant $1/\omega$ singularity of our on-shell Z_2 becomes apparent when one compares S-matrix elements of QCD and EFT. With N_E external heavy fermions, these differ from truncated MS-renormalized Green functions by factors of $(Z_2^{MS}(\mu)/Z_2)^{N_E/2}$ and $(\tilde{Z}_2^{MS}(\mu)/\tilde{Z}_2)^{N_E/2}$ respectively. Now the Green functions are finite, by construction, and the S-matrix elements of the two theories can differ, at most, by finite radiative corrections which vanish as $M \rightarrow \infty$ and hence $\bar{\alpha}_s(M) \rightarrow 0$. Moreover $\tilde{Z}_2 = 1$, since there can be no on-shell wave-function renormalization in dimensionally regularized EFT, as all the integrals contributing to the on-shell self energy are scale free. It follows that all singularities must cancel in the finite ratio $R(\mu) \equiv (Z_2^{MS}(\mu)/Z_2)/(\tilde{Z}_2^{MS}(\mu)/\tilde{Z}_2)$ and hence that a knowledge of Z_2 suffices to determine the difference of QCD and EFT anomalous dimensions. In Sect. 4 we shall verify that this is indeed the case at the two-loop level (provided one neglects heavy-quark loops in QCD, since these are discarded *ab initio* in EFT).

It is thus apparent that the gauge invariance of Z_2 guarantees the gauge invariance of the difference (11) of the anomalous dimensions of QCD and EFT. It was long ago remarked [24] that to leading order Z_2 has no ultraviolet divergence in the Landau gauge and no infrared divergence in the Yennie gauge. Dimensional regularization assigns Z_2 a unique gauge-invariant value in QED and (to two loops, at least) in QCD. Since this unique value provides an important link between QCD and EFT, its calculation becomes of practical as well as theoretical interest.

2.3 Two-loop result

We now need the two-loop integrals contributing to

$$M_2 = -A_2 + (D-2)A_1^2 \quad (12)$$

$$F_2 = B_2 - 4\omega A_2 - 2A_2' + \left(\frac{D^2 - 7D + 8 + a_0}{D-5} \right) A_1^2. \quad (13)$$

In [7] we gave the exact result for the two-loop term A_2 , required in (12). It involved four colour factors and the three terms

$$R_1 = F^2(\omega), \quad R_2 = \frac{\omega \Gamma^2(-\omega) \Gamma(-4\omega) \Gamma(2\omega) \Gamma(\omega)}{\Gamma(-2\omega) \Gamma(-3\omega)}, \quad (14)$$

$$R_3 = I(\omega)$$

which derive from the three irreducible integrals to which all other on-shell two-loop integrals may eventually be reduced by the method of integration by parts [7]. The last of these is the D -dimensional (Minkowski space) integral

$$I(\omega) \equiv \frac{(p^2)^{5-D}}{\pi^D} \iint \frac{d^D k d^D l}{(k^2 + 2p \cdot k) k^2 (l^2 + 2p \cdot l) l^2 ((k+l)^2 + 2p \cdot (k+l))} \quad (15)$$

$$= \pi^2 \log 2 - \frac{3}{2} \zeta(3) + O(\omega) \quad (16)$$

whose 4-dimensional value was obtained in [17].

In this paper, we find it convenient to work with the colour factors

$$C_1 = C_F(C_A - 2C_F), \quad C_2 = C_F^2, \quad C_3 = 2T_F N_L C_F, \quad (17)$$

$$C_4 = 2T_F C_F$$

where $C_A = N_C$ and $T_F = \frac{1}{2}$ for a gauge group $SU(N_C)$ and N_L is the number of light fermions contributing to Fig. 1d, here taken to be massless. Note that Fig. 1b, c give gauge-dependent contributions proportional to the colour factors C_1 and C_2 , respectively, whilst the light- and heavy-quark loops in Fig. 1d give gauge-invariant contributions proportional to C_3 and C_4 , respectively. The nonabelian couplings in Fig. 1c, f and the ghost loop in Fig. 1g give gauge-dependent contributions proportional to $C_F C_A = C_1 + 2C_2$. In the case of one-lepton QED, one sets $C_A = N_L = 0$ and $C_F = T_F = 1$.

In terms of the structures (14) and (17) the two-loop coefficient M_2 of Z_m in (3) is given by Table 1, which lists the non-vanishing coefficients M_{ij} of the matrix coupling the colour and integral structures in

$$M_2 = \sum_{i=1}^4 \sum_{j=1}^3 C_i M_{ij} R_j. \quad (18)$$

For the $O(g_0^4)$ corrections to Z_2 we need to calculate new two-loop terms, namely the B_2 and A_2' terms of (13). These may be obtained by the methods of [7], albeit with considerably greater effort, needed to extend the recurrence relations to deal with terms which are generated by the doubling of fermion propagators in A_2' . We have evaluated them for any dimension, D , and gauge parameter, a_0 , but the results are too bulky to reproduce here. What concerns us is the combination (13), which turns out to be gauge-invariant, thanks to remarkable cancellations, between diagrams, of terms linear and quadratic in a_0 in several (colour factor \times integral) structures, each of which involves complicated rational functions of D , of which Table 1 is indicative. Since we are

Table 1. Non-vanishing coefficients M_{ij} of $C_i R_j$ in (18)

$M_{11} =$	$\frac{3(5D^3 - 58D^2 + 180D - 152)}{2(3D-8)(3D-10)(D-3)}$
$M_{12} =$	$\frac{4(4D^3 - 41D^2 + 122D - 104)}{3(3D-8)(3D-10)(D-3)}$
$M_{13} =$	$-\frac{4(D^2 - 7D + 8)(D-3)(D-6)}{(3D-8)(3D-10)}$
$M_{21} =$	$\frac{D^2 - 8D + 13}{(D-3)^2}$
$M_{22} =$	$-\frac{8(2D^3 - 21D^2 + 68D - 71)(D-2)}{3(3D-8)(3D-10)(D-3)^2}$
$M_{32} =$	$-\frac{16(D-2)}{3(3D-8)(3D-10)}$
$M_{41} =$	$-\frac{12(D^3 - 12D^2 + 50D - 68)}{(3D-8)(3D-10)(D-3)(D-6)}$
$M_{42} =$	$\frac{16(D^3 - 7D^2 + 6D + 16)}{3(3D-8)(3D-10)(D-3)(D-6)}$
$M_{43} =$	$-\frac{8(D^3 - 7D^2 + 6D + 16)(D-4)}{(3D-8)(3D-10)(D-6)}$

Table 2. Non-vanishing coefficients F_{ij} of $C_i R_j$ in (19)

$F_{11} =$	$-\frac{3D^5 - 61D^4 + 469D^3 - 1679D^2 + 2756D - 1648}{8(D-3)^2(D-5)^2}$
$F_{12} =$	$-\frac{2D^5 - 29D^4 + 148D^3 - 321D^2 + 268D - 60}{3(3D-10)(D-3)^2(D-5)}$
$F_{21} =$	$-\frac{D^3 - 12D^2 + 37D - 36}{4(D-3)^2}$
$F_{22} =$	$\frac{2(2D^3 - 17D^2 + 42D - 29)(D-2)}{3(3D-10)(D-3)^2}$
$F_{32} =$	$\frac{4(D-2)}{3(3D-10)}$
$F_{41} =$	$-\frac{2(D^2 - 8D + 11)(D-4)}{(D-2)(D-3)(D-5)(D-7)}$

now highly sensitive to the three-gluon coupling of a non-abelian gauge theory, we regard the two-loop gauge invariance of Z_2 as a strong indication of its gauge invariance to *all* orders. It should however be remarked that we are not yet sensitive to the four-gluon coupling.

To present our result compactly, we exploit another interesting feature, namely that the combination

$$F_2 - (1 + D/4) M_2 = \sum_{i=1}^4 \sum_{j=1}^2 C_i F_{ij} R_j \quad (19)$$

does not involve the integral (15). As in the case of the one-loop relationship $F_1 = M_1$, we lack an argument as to why such a simplification should occur. It involves matching cancellations in each of Fig. 1b, d and these are apparent only after extensive use of the recurrence relations of [7]. Based on these two instances, one is tempted to speculate that at L loops there is always a linear combination of F_L and M_L in which there is

no net contribution from intermediate states with the maximum number of massive fermions, namely $2L-1$. The proof of such a conjecture might be easier to find in old-fashioned, time-ordered perturbation theory.

Thanks to the relative simplicity of combination (19) and to gauge invariance, we are able to give a complete account of two-loop on-shell fermion mass and wave-function renormalization, in any dimension D , by complementing Table 1 with Table 2. In comparison to individual results for the contribution of a particular diagram to one of the relevant terms $\{A_2, A'_2, B_2\}$, the full D -dimensional results of Tables 1 and 2 are rather compact.

2.4 Laurent expansion as $D \rightarrow 4$

We now perform Laurent expansions in ω , obtaining the following two-loop results, in terms of the *bare* coupling:

$$Z_m = 1 - \left(\frac{\alpha_0}{\pi M^2 \omega} \right) C_F \left\{ \frac{3}{4\omega} + 1 + \left(\frac{3}{8} \zeta(2) + 2 \right) \omega + O(\omega^2) \right\} + \left(\frac{\alpha_0}{\pi M^2 \omega} \right)^2 \cdot \sum_{i=1}^4 C_i \{ M_2^i / \omega^2 + M_1^i / \omega + M_0^i + O(\omega) \} + O(\alpha_0^3) \quad (20)$$

$$Z_2 = 1 - \left(\frac{\alpha_0}{\pi M^2 \omega} \right) C_F \left\{ \frac{3}{4\omega} + 1 + \left(\frac{3}{8} \zeta(2) + 2 \right) \omega + O(\omega^2) \right\} + \left(\frac{\alpha_0}{\pi M^2 \omega} \right)^2 \cdot \sum_{i=1}^4 C_i \{ F_2^i / \omega^2 + F_1^i / \omega + F_0^i + O(\omega) \} + O(\alpha_0^3) \quad (21)$$

where $\alpha_0 \equiv (g_0^2/4\pi)(4\pi/e^{\gamma})^\omega$ and the two-loop coefficients M_n^i and F_n^i , associated with the colour factors (17), are given in Table 3. Note that it is necessary to retain the one-loop $O(\omega)$ terms, since these generate finite contributions after coupling constant renormalization.

In [7], we used (20) to derive the relation between the pole mass and the three-loop MS mass. In Sects. 4 and 5 we apply (21) to wave-function renormalization in different schemes of coupling constant renormalization.

Table 3. Coefficients M_n^i and F_n^i of C_i/ω^n in (20) and (21)

n	2	1	0
M_n^1	$-\frac{1}{32}$	$-\frac{9}{64}$	$\frac{3}{32} \zeta(2) - \frac{1}{4} I(0) - \frac{595}{128}$
M_n^2	$-\frac{1}{32}$	$-\frac{13}{64}$	$-\frac{1}{32} \zeta(2) - \frac{101}{128}$
M_n^3	$\frac{1}{16}$	$\frac{7}{32}$	$\frac{5}{16} \zeta(2) + \frac{35}{64}$
M_n^4	$\frac{1}{16}$	$\frac{7}{32}$	$-\frac{7}{16} \zeta(2) + \frac{59}{64}$
F_n^1	$-\frac{1}{32}$	$-\frac{101}{64}$	$\frac{49}{32} \zeta(2) - \frac{1}{4} I(0) - \frac{803}{128}$
F_n^2	$-\frac{1}{32}$	$-\frac{151}{64}$	$-\frac{49}{32} \zeta(2) - \frac{1173}{128}$
F_n^3	$\frac{1}{16}$	$\frac{9}{32}$	$\frac{5}{16} \zeta(2) + \frac{59}{64}$
F_n^4	$\frac{1}{8}$	$\frac{19}{96}$	$-\frac{7}{8} \zeta(2) + \frac{1139}{96}$

tion, namely minimal subtraction of the QCD coupling, and on-shell charge renormalization in QED. But first we calculate the form of the contribution of Fig. 1d when the internal fermion is neither massless, nor of the external mass M , since this is clearly of some consequence in QED, where the effects of one of the leptons $\{e, \mu, \tau\}$ on the other two need investigation.

3 Radiative effects of non-trivial fermion mass ratios

The effect on (20) of finite internal fermion mass $M_i \neq M$ in Fig. 1d was computed, in terms of dilogarithms, in [7]. The same dilogarithms suffice to express the corresponding effect on (21), but in the case of wave-function renormalization they result from a finite integral over the fermion contribution to the gauge-boson propagator subtracted at zero momentum. This is because we must separate out infrared singularities present in Z_2 , but absent from Z_m . We find that the $O(g_0^4)$ contribution to Z_2 , of a single internal fermion of mass $M_i = rM$, is of the form

$$\Delta Z_2 = \left(\frac{\alpha_0}{\pi M^2 \omega} \right)^2 C_4 \left(\frac{1}{8\omega^2} + \frac{19 - 24 \log r}{96\omega} + \frac{1}{4} \log^2 r - \frac{1}{24} \log r + \frac{1}{8} \zeta(2) + \frac{59}{192} + \bar{\Delta}(r) + O(\omega) \right) \quad (22)$$

where

$$\bar{\Delta}(r) = \frac{1}{8} \int_0^1 \frac{dy}{y} (2+y)(1-y) \bar{\Pi}(r^2(1-y)/y^2)$$

$$\bar{\Pi}(z) = 2(1-2z) \sqrt{1+4z} \operatorname{arccoth} \sqrt{1+4z+4z} - \frac{5}{3}$$

and the bars are to distinguish $\bar{\Delta}$ and $\bar{\Pi}$ from the related but different functions Δ and Π involved in the corresponding analysis [7] of Z_m . Note the presence of a mass-dependent singular term, $\omega^{-1} \log r$, in (22). In Sect. 5 we will show that this is removed by on-shell charge renormalization of QED.

It remains to reduce $\bar{\Delta}(r)$ to the dilogarithms [7]

$$\begin{aligned} L_{\pm}(r) &\equiv \int_0^1 dx \left(\frac{\log x - \log r}{x \pm r} \right) \\ &= \frac{1}{2} \log^2 r + \left(\frac{1}{2} \mp \frac{3}{2} \right) \zeta(2) - L_{\pm}(1/r) \\ &= \log r \log \left(\frac{r}{r \pm 1} \right) + \operatorname{Li}_2 \left(\mp \frac{1}{r} \right) \quad \text{for } r \geq 1 \end{aligned}$$

where $\operatorname{Li}_p(x) = \sum_{n=1}^{\infty} x^n/n^p$ for $p > 1 \geq |x|$. An intricate calculation yields

$$\begin{aligned} \bar{\Delta}(r) &= \frac{1}{8}(r+1)(6r^3 - r^2 - r + 2) L_{-}(r) \\ &\quad + \frac{1}{8}(r-1)(6r^3 + r^2 + r - 2) L_{-}(r) \\ &\quad + \frac{1}{24} \log^2 r + \frac{229}{288} + \left(\frac{1}{4} \log r + \frac{7}{8} \right) r^2 \\ &= \sum_{n=1}^{\infty} (-2G(n) \log r + G'(n)) r^{-2n} \quad \text{for } r \geq 1 \quad (23) \end{aligned}$$

where $G(n) = 3(n^2 - 1)/4n(n+2)(2n+1)(2n+3)$ and $G'(n)$ is its derivative.

A check on this result is provided by setting $r=1$. We find that $\bar{\Delta}(1) = \frac{481}{288} - \zeta(2)$, giving a contribution (22) which, with $r=1$, agrees with the C_4 term of (21). This agreement between a long algebraic calculation and a difficult analytical evaluation gives us considerable confidence in each. The limiting behaviours of $\bar{\Delta}(r)$ at large and small mass ratios r are as follows:

$$\bar{\Delta}(r) = \frac{1}{30} r^{-2} + O(r^{-4} \log r) \quad (24)$$

$$\bar{\Delta}(r) = \frac{1}{4} \log^2 r + \frac{1}{24} \log r + \frac{1}{4} \zeta(2) + \frac{229}{288} + O(r) \quad (25)$$

in marked contrast to the corresponding term $\Delta(r)$ in

$$\begin{aligned} \Delta Z_m &= \left(\frac{\alpha_0}{\pi M^2 \omega} \right)^2 C_4 \left(\frac{1}{16\omega^2} + \frac{7}{32\omega} + \frac{5}{16} \zeta(2) + \frac{45}{64} - \Delta(r) \right. \\ &\quad \left. + O(\omega) \right) \quad (26) \end{aligned}$$

which is given exactly by [7]

$$\begin{aligned} \Delta(r) &= -\frac{1}{4}(r+1)(r^3+1) L_{-}(r) - \frac{1}{4}(r-1)(r^3-1) L_{-}(r) \\ &\quad + \frac{1}{4} \log^2 r + \frac{1}{4} \zeta(2) - \left(\frac{1}{4} \log r + \frac{3}{8} \right) r^2 \quad (27) \end{aligned}$$

and has the limiting behaviours

$$\Delta(r) = \frac{1}{4} \log^2 r + \frac{1}{24} \log r + \frac{1}{4} \zeta(2) + \frac{151}{288} + O(r^{-2} \log r) \quad (28)$$

$$\Delta(r) = \frac{3}{4} \zeta(2) r + O(r^2) \quad (29)$$

with $\Delta(1) = \frac{3}{4} \zeta(2) - \frac{3}{8}$.

To summarize thus far: in Sect. 2 we found the contribution of N_f massless fermions and the fermion of mass M to Fig. 1d in any dimension D , whilst in this section we deal with internal fermions of any finite mass, but must resort to dilogarithms to find their contributions as $D \rightarrow 4$. This complication does not affect the proof of the gauge invariance of Z_2 to two loops in all dimensions, since Fig. 1d is separately gauge-invariant, for any fermion mass ratio. It is, however, apparent from (22, 25) that for Z_2 (unlike Z_m) one must decide *ab initio* whether one treats light quarks as massless: there is clearly no way of obtaining the massless quark contributions from those of finite-mass quarks, since the vanishing of r in (22) produces infrared mass singularities, which were dimensionally regularized in Sect. 2. Despite this complication, we have sufficient equations to handle all mass cases and may now proceed to renormalize the coupling.

4 MS coupling renormalization in QCD and EFT

In QCD, unlike QED, one cannot renormalize the coupling merely by calculating the wave-function renormalization of the gauge boson on its $q^2=0$ mass shell: that is the really significant consequence of the nonabelian structure. Our perturbative analysis suggests that the on-shell infrared problems of quarks and leptons are rather similar and equally gauge-invariant, after dimensionally

regularized on-shell fermion wave-function renormalization. But gluons are decidedly different from photons, even in perturbation theory. This, we suggest, is the *real* infrared problem of QCD: gluon confinement. If so, there is hope of devising an intermediate scheme, in which one dares to approach the perturbative heavy-quark mass shell, but requires substantial virtualities of gluons and light quarks, which dress the heavy quark as a decent hadron. As stressed in a recent review by Bjorken [29], the formal limit $M \rightarrow \infty$ of EFT [13, 14] provides a well-defined starting point for such an attempt.

4.1 Derivation of $\tilde{\gamma}_F$ from Z_2

To relate our result for Z_2 to EFT, we renormalize the coupling in the MS scheme, with N_L light quarks:

$$\frac{g_0^2}{4\pi} = \left(\frac{\mu^2 e^\gamma}{4\pi}\right)^\omega \tilde{\alpha}_s(\mu) \left(1 - \frac{\tilde{\alpha}_s(\mu)}{\pi\omega} \left(\frac{1}{12} C_A - \frac{1}{3} T_F N_L\right) + O(\tilde{\alpha}_s^2)\right) \quad (30)$$

where μ is an arbitrary mass scale, introduced to make $\tilde{\alpha}_s$ dimensionless, and the power of $(e^\gamma/4\pi)$ suppresses needless factors of $(\log 4\pi - \gamma)$ in the $\omega \rightarrow 0$ limit of (20, 21). It is important to realize that (30) applies for *all* $D = 4 - 2\omega$; not just as $\omega \rightarrow 0$. There are no further terms in the Laurent expansion, otherwise the renormalization would not be minimal. Note also that we do not include the effect of the heavy-quark loop in (30), since that is discarded in EFT.

After MS coupling renormalization, the Laurent expansion (21) may conveniently be decomposed as

$$Z_2 = Z_2^L + Z_2^H + O(\tilde{\alpha}_s^3) \quad (31)$$

where

$$Z_2^L = 1 - \left(\frac{\alpha_s(M)}{\pi}\right) C_F \left\{ \frac{3}{4\omega} + 1 + \left(\frac{3}{8}\zeta(2) + 2\right)\omega + O(\omega^2) \right\} + \left(\frac{\alpha_s(M)}{\pi}\right)^2 \sum_{i=1}^3 C_i \{ \bar{F}_2^i/\omega^2 + \bar{F}_1^i/\omega + \bar{F}_0^i + O(\omega) \} \quad (32)$$

is the contribution of light quarks and gluons, whilst

$$Z_2^H = \left(\frac{\alpha_s(M)}{\pi}\right)^2 C_4 \left(\frac{1}{8\omega^2} + \frac{19}{96\omega} - \frac{7}{8}\zeta(2) + \frac{1139}{576} + O(\omega) \right) \quad (33)$$

is the contribution of the heavy quark itself, which is unaffected by coupling renormalization and will play no role in establishing the link with EFT.

The coefficients \bar{F}_n^i of Table 4 are obtained from the corresponding coefficients in Table 3, taking into account the renormalization of the one-loop term by (30). They uniquely determine the minimal subtractions in

$$\frac{Z_2^{\text{MS}}(\mu)}{\tilde{Z}_2^{\text{MS}}(\mu)} = 1 - 3 C_F \left(\frac{\tilde{\alpha}_s(\mu)}{4\pi\omega} \right) + C_F \left(\frac{\tilde{\alpha}_s(\mu)}{4\pi\omega} \right)^2 \cdot \left\{ \frac{1}{12} C_A + \frac{9}{2} C_F - 2 T_F N_L - \left(\frac{1}{12} C_A - \frac{1}{3} C_F - \frac{1}{3} T_F N_L \right) \omega \right\} + O(\tilde{\alpha}_s^3) \quad (34)$$

Table 4. MS-renormalized coefficients \bar{F}_n^i of C_i/ω^n in (32)

n	2	1	0
\bar{F}_n^1	$\frac{1}{32}$	$-\frac{127}{192}$	$\frac{1}{8}\zeta(2) - \frac{1}{4}I(0) - \frac{1705}{384}$
\bar{F}_n^2	$\frac{3}{32}$	$-\frac{191}{192}$	$-\frac{27}{32}\zeta(2) - \frac{2111}{384}$
\bar{F}_n^3	$-\frac{1}{16}$	$\frac{1}{96}$	$\frac{1}{4}\zeta(2) + \frac{113}{192}$

by the requirement that $R(\mu) \equiv (Z_2^{\text{MS}}(\mu)/Z_2^L)/\tilde{Z}_2^{\text{MS}}(\mu)$ be finite as $\omega \rightarrow 0$. Note that $Z_2^{\text{MS}}(\mu)/\tilde{Z}_2^{\text{MS}}(\mu)$ is *not* obtained by mere subtraction of the singularities in Z_2^L , but rather by the requirement that (34) have a minimal structure such that when divided by the non-minimal \tilde{Z}_2^L the result, $R(\mu)$, is finite. The finiteness of $R(\mu)$ then ensures that a ratio of QCD and EFT S-matrix elements is finite, given that the corresponding ratio of renormalized Green functions is finite and that there is no on-shell wave-function renormalization in dimensionally regularized EFT.

A strong check on (34) is provided by calculating the difference of the anomalous dimensions (8) and (10), using the D -dimensional beta function

$$\frac{d \log \tilde{\alpha}_s(\mu)}{d \log \mu} = -2\omega - 2 \frac{\tilde{\alpha}_s(\mu)}{\pi} \left(\frac{1}{12} C_A - \frac{1}{3} T_F N_L \right) + O(\tilde{\alpha}_s^2)$$

which gives the finite result

$$\gamma_F - \tilde{\gamma}_F = \frac{3 C_F \tilde{\alpha}_s}{2\pi} + \left(\frac{127}{192} C_A - \frac{1}{3} C_F - \frac{1}{3} T_F N_L \right) \frac{C_F \tilde{\alpha}_s^2}{4\pi^2} + O(\tilde{\alpha}_s^3) \quad (35)$$

$$= 2 \frac{\tilde{\alpha}_s}{\pi} + \left(\frac{4}{3} - \frac{1}{18} N_L \right) \frac{\tilde{\alpha}_s^2}{\pi^2} + O(\tilde{\alpha}_s^3) \quad \text{for SU(3).} \quad (36)$$

Combining (35) with the known [20] two-loop QCD anomalous dimension

$$\gamma_F = \frac{\bar{a} C_F \tilde{\alpha}_s}{2\pi} + \left\{ \left(\frac{\bar{a}^2}{32} + \frac{\bar{a}}{4} + \frac{25}{32} \right) C_A - \frac{3}{16} C_F - \frac{1}{4} T_F N_L \right\} \frac{C_F \tilde{\alpha}_s^2}{\pi^2} + O(\tilde{\alpha}_s^3) \quad (37)$$

we obtain the EFT result

$$\tilde{\gamma}_F = \frac{(\bar{a} - 3) C_F \tilde{\alpha}_s}{2\pi} + \left\{ \left(\frac{\bar{a}^2}{32} + \frac{\bar{a}}{4} - \frac{179}{96} \right) C_A + \frac{2}{3} T_F N_L \right\} \frac{C_F \tilde{\alpha}_s^2}{\pi^2} + O(\tilde{\alpha}_s^3) \quad (38)$$

which has recently been verified by Broadhurst and Grozin [16], working entirely within EFT. Note that the effective field theory obtained by taking the electron mass to infinity in pure QED corresponds to $C_A = N_L = 0$ and hence has no anomalous dimension at two loops in the (renormalized) Yennie gauge, which was chosen for precisely that reason in [27]. By contrast, the EFT of a static quark is not greatly simplified by choosing the Yennie gauge, since there is still an anomalous dimension at the two-loop level.

4.2 Renormalization-group improvement

We can integrate (36), using the one- and two-loop terms of the beta function [26]

$$2\omega + \frac{d \log \bar{\alpha}_s}{d \log \mu} = -2 \sum_{n=1}^{\infty} b_n \left(\frac{\bar{\alpha}_s}{4\pi} \right)^n$$

with $b_1 = 11 - \frac{2}{3}N_L$ and $b_2 = 102 - \frac{38}{3}N_L$. Writing (36) in the similar form

$$\frac{d \log(Z_2^{\text{MS}}/\bar{Z}_2^{\text{MS}})}{d \log \mu} = 2 \sum_{n=1}^{\infty} e_n \left(\frac{\bar{\alpha}_s}{4\pi} \right)^n$$

with $e_1 = 4$ and $e_2 = 82 - \frac{44}{3}N_L$, we readily obtain the 4-dimensional, two-loop, renormalization-group improved result

$$R(\mu) \equiv \frac{1}{Z_2^1} \frac{Z_2^{\text{MS}}(\mu)}{\bar{Z}_2^{\text{MS}}(\mu)} \approx R(M) \left(\frac{\bar{\alpha}_s(M)}{\bar{\alpha}_s(\mu)} \right)^{4/b_1} \left(\frac{1 + E_2 \bar{\alpha}_s(M)/\pi}{1 + E_2 \bar{\alpha}_s(\mu)/\pi} \right) \quad (39)$$

$$E_2 \equiv \frac{e_2 b_1 - e_1 b_2}{4b_1^2} = \frac{175}{162} \quad \text{or} \quad \frac{4253}{3750} \quad \text{for } N_L = 3 \quad \text{or} \quad 4 \quad (40)$$

for the finite ratio (39) of the factors which convert MS-renormalized Green functions to S-matrix elements in QCD and EFT. Moreover, the finite part of (32) determines the integration constant $R(M)$ in (39) to be

$$R(M) = 1 + \frac{4}{3} \bar{\alpha}_s(M)/\pi + K_2 \bar{\alpha}_s^2(M)/\pi^2 + O(\bar{\alpha}_s^3) \quad (41)$$

$$K_2 = \frac{2}{9} \pi^2 \log 2 - \frac{1}{3} \zeta(3) + \frac{7}{6} \zeta(2) + \frac{4663}{288} - \left(\frac{1}{3} \zeta(2) + \frac{113}{144} \right) N_L \approx 19.23 - 1.33 N_L. \quad (42)$$

Thus, from the gauge-invariant, renormalization-group invariant, on-shell quantity (32) we have derived the renormalization-group improved, two-loop expression (39) for the scale-dependence of the ratio of two gauge-dependent artifacts of the MS scheme and also the boundary condition (41) to the level commensurate with three-loop MS renormalization.

It is clear that on-shell wave-function renormalization corresponds in many respects with on-shell mass [7] renormalization: each is gauge-invariant; each determines a gauge-invariant anomalous dimension; the finite parts of each at two loops are needed to relate off-shell results of the MS scheme at three loops to physical quantities; these finite parts are large. For comparison with (42), note that the corresponding coefficient of $\bar{\alpha}_s^2(M)/\pi^2$ in $Z_m^{\text{MS}}(M)/Z_m = M/\bar{m}(M)$ is $K = 16.11 - 1.04 N_L$ [7].

Finally, we remark on the relation between the leading behaviour of (32) and the one-loop EFT anomalous dimension $\tilde{\gamma}_1$ of heavy-light $\bar{Q} \gamma_\mu (\gamma_5) q$ currents, apparent in the logarithms of [15] and elucidated in [14]. From our point of view, it is best obtained from the gauge-invariant one-loop dimensionally regularized singularity of (32), associated with an on-shell fermion:

$$\tilde{\gamma}_1 = \frac{1}{2} (\tilde{\gamma}_F - \gamma_F) + O(\bar{\alpha}_s^2) = -\bar{\alpha}_s/\pi + O(\bar{\alpha}_s^2) = -\frac{1}{2} \gamma_m + O(\bar{\alpha}_s^2). \quad (43)$$

In the Landau gauge, one may blame it all on the static-quark field, since the coupling and the light-quark field are regular:

$$\tilde{\gamma}_1 = \frac{1}{2} \tilde{\gamma}_F(\bar{a}=0) + O(\bar{\alpha}_s^2).$$

In the Yennie gauge, the static-quark field is regular, but the divergence of the coupling has the opposite sign to that of the light-quark field and twice its magnitude, since the light-light $\bar{q} \gamma_\mu (\gamma_5) q$ current is conserved:

$$\tilde{\gamma}_1 = (\frac{1}{2} - 1) \gamma_F(\bar{a}=3) + O(\bar{\alpha}_s^2).$$

The $O(\bar{\alpha}_s^2)$ corrections to the relations between the anomalous dimensions of (43) are studied in detail in [16], in an arbitrary covariant gauge.

5 Complete on-shell two-loop renormalization of QED

We achieve this in three stages. First we give exact results, in D dimensions, for all the two-loop renormalization constants of 'pure' QED, uncomplicated by electroweak effects or the existence of μ and τ . In other words, we effect the two-loop on-shell renormalization of the $U(1)$ gauge theory of a single fermion in D dimensions. Then we give the Laurent expansions of the renormalization constants, including finite parts. Finally we indicate how these are modified by the addition of other leptons. We take no account of the existence of weak interactions.

5.1 D -dimensional QED, without integration

There is only one more independent renormalization constant to determine in QED: the on-shell photon wave-function renormalization constant Z_3 , which also determines the charge renormalization $e_0^2 = e_R^2/Z_3$, thanks to the Ward identity [30] $Z_1 = Z_2$.

In comparison with Z_m and Z_2 , we find it rather easy to calculate $Z_3 = 1/(1 + \Pi(0))$, to two loops, from the bare-photon self energy $\Pi(q^2)$ at $q^2 = 0$. One has merely to operate on self-energy diagrams with $(\partial^2/\partial q_\alpha \partial q_\beta)$ and then set the external momentum q to zero. This results in a series of bubble diagrams, with four insertions of gamma matrices, which add up to give $\Pi(0)$ times the constant tensor

$$(\partial^2/\partial q_\alpha \partial q_\beta)(q_\mu q_\nu - q^2 g_{\mu\nu}) = g_{\mu\alpha} g_{\nu\beta} + g_{\nu\alpha} g_{\mu\beta} - 2 g_{\mu\nu} g_{\alpha\beta}.$$

The one-loop integrals give a multiple of $\Gamma(\omega)$, along with obvious powers of π , e_0 and m_0 . Very conveniently, every two-loop integral [7] gives a rational function of D times $\Gamma^2(\omega)$. It is thus a simple matter of book-keeping to obtain the two-loop expansion in terms of the bare quantities and then use the one-loop renormalization of e_0 and m_0 to express Z_3 in terms of the physical charge e_R and physical mass M in any dimension D . A short REDUCE program yields

$$\frac{e_R^2}{e_0^2} = Z_3 = 1 - \frac{4}{3} \left(\frac{e_R^2 \Gamma(\omega)}{(4\pi)^{D/2} M^{2\omega}} \right) + B(D) \left(\frac{e_R^2 \Gamma(\omega)}{(4\pi)^{D/2} M^{2\omega}} \right)^2 + O(e_R^6) \quad (44)$$

where e_R is the D -dimensional physical coupling constant, measured at zero momentum, and

$$B(D) = -\frac{2(D-4)}{D(D-3)(D-5)} \{2 + (D-4)(D^2 - 8D + 9)\}. \quad (45)$$

The simple rationality of (45) belies its power. It determines not only how the two-loop coupling of any off-shell scheme runs, but also the boundary condition for the integral solution to the renormalization-group equation for the running coupling. The former information is encoded by the leading behaviour as $D \rightarrow 4$: $B'(4) = 1$; the latter by the next-to-leading behaviour: $B''(4) = -15/2$. Merely by manipulating gamma matrices and gamma functions in D -dimensions at zero momentum, we obtain these two crucial numbers, which require the running coupling $\bar{\alpha}(\mu)$ of the MS scheme to satisfy

$$\frac{\pi}{\bar{\alpha}(\mu)} = \frac{\pi}{\alpha} + \frac{2}{3} \log \frac{M}{\mu} + \frac{\alpha}{\pi} \left(\frac{1}{2} \log \frac{M}{\mu} - \frac{15}{16} \right) + O(\alpha^2) + O(\omega) \quad (46)$$

where $\alpha \equiv \lim_{D \rightarrow 4} e_R^2/4\pi$ is the fine structure constant, as measured in 4 dimensions. To obtain (46), one has merely to equate the D -dimensional MS ansatz

$$\frac{e_0^2}{4\pi} = \left(\frac{\mu^2 e^\gamma}{4\pi} \right)^\omega \bar{\alpha}(\mu) \left(1 + \frac{\bar{\alpha}(\mu)}{\pi} \frac{Z_{11}}{\omega} + \frac{\bar{\alpha}^2(\mu)}{\pi^2} \left(\frac{Z_{22}}{\omega^2} + \frac{Z_{21}}{\omega} \right) + O(\bar{\alpha}^3) \right) \quad (47)$$

to $Z_3^{-1} e_R^2/4\pi$ and require that $\bar{\alpha}(\mu)$ be finite in 4 dimensions. This physical constraint on the MS scheme yields the subtraction constants of (47) and the solution (46) to the renormalization-group equation [31]

$$\beta(\bar{\alpha}(\mu)) \equiv 2\omega + \frac{d \log \bar{\alpha}(\mu)}{d \log \mu} = \frac{2}{3} \frac{\bar{\alpha}(\mu)}{\pi} + \frac{1}{2} \frac{\bar{\alpha}^2(\mu)}{\pi^2} + O(\bar{\alpha}^3). \quad (48)$$

Thus the on-shell Z_3 contains, in its finite part, more information than can be obtained by ultraviolet subtraction: it tells the QED MS coupling where to run to, in order to agree with on-shell data, rather than leaving it with an integration constant like the astronomic value of $A_{\text{QED}} \sim M \exp(3\pi/2\alpha)$ [31, 33]. This finite information is as easy to obtain from (44, 45) as is the beta function.

Lest it be thought that this virtue of on-shell renormalization is peculiar to the infrared freedom of QED, we remark that an analogous situation arose concerning the relationship between the pole and MS masses of heavy quarks in QCD [7]. There one was in the ironic situation of knowing the three-loop anomalous dimension γ_m [11], but being unable to use it to relate constituent and current quark masses, for lack of the finite two-loop part of Z_m . This state of affairs was remedied in

[7], where it was shown that the finite on-shell two-loop term dominates the next-to-leading corrections.

These two examples show the utility of obtaining on-shell renormalization constants in D -dimensions, in order to extract physically relevant finite parts, as well as anomalous dimensions. We therefore give a complete description of the on-shell two-loop renormalization of QED in any dimension by complementing (44) with the corresponding expansions of Z_m and Z_2 in terms of the physical charge and mass:

$$Z_m = 1 - \frac{D-1}{D-3} \left(\frac{e_R^2 \Gamma(\omega)}{(4\pi)^{D/2} M^{2\omega}} \right) + \sum_{j=1}^3 M_j(D) \frac{R_j}{R_1} \left(\frac{e_R^2 \Gamma(\omega)}{(4\pi)^{D/2} M^{2\omega}} \right)^2 + O(e_R^6) \quad (49)$$

$$Z_2 = 1 - \frac{D-1}{D-3} \left(\frac{e_R^2 \Gamma(\omega)}{(4\pi)^{D/2} M^{2\omega}} \right) + \sum_{j=1}^3 F_j(D) \frac{R_j}{R_1} \left(\frac{e_R^2 \Gamma(\omega)}{(4\pi)^{D/2} M^{2\omega}} \right)^2 + O(e_R^6) \quad (50)$$

where the rational functions multiplying the integral structures (14) are obtained from the coefficients of Tables 1 and 2 as follows:

$$M_j(D) = -2M_{1j} + M_{2j} + 2M_{4j} - \frac{4(D-1)}{3(D-3)} \delta_{j1} \quad (51)$$

$$F_j(D) = -2F_{1j} + F_{2j} + 2F_{4j} + \frac{D(D-1)}{3(D-3)} \delta_{j1} + (1 + D/4) M_j(D) \quad (52)$$

by setting $C_A = N_L = 0$ and $C_F = T_F = 1$ in (17) and using (44) to transform to the physical charge. The explicit forms of these coefficients involve polynomials in D of orders up to 10. Their Laurent expansions are used in the next section.

We remark that the rationality of D -dimensional calculation extends beyond the calculation of renormalization constants. It is clear that the two-loop anomalous magnetic moment calculation involves only zero-momentum insertions in Fig. 1a, b, d, e. after differentiating with respect to an infinitesimal external photon momentum. Thus $g-2$ to two loops, in D -dimensions, can likewise be reduced to the same three integral structures, by systematic computer algebra, quite free of anything remotely resembling integration over Feynman or Schwinger parameters.

Nor does the avoidance of integration end here, since one of us has found [19] that the sole recalcitrant integral, $I(\omega)$, may be reduced, in any dimension, to Γ functions and a single Saalschützian ${}_3F_2$ series, whose power expansion in ω can be found up to the level required for four-loop calculations by a combination of finite group theory and known special cases of related series, mainly culled from Hardy's lucid exegesis [32] of Chapter XII of Ramanujan's notebook. This expansion involves $\{\text{Li}_p(1), \text{Li}_p(\frac{1}{2}) | p \leq 5\}$, yet no Spence integral is ever encountered; computer algebra suffices.

We defer consideration of $g-2$ and higher-order terms in $I(\omega)$ to subsequent papers, here making the general point that, by mere book-keeping in D dimensions, much may be calculated which previously appeared to entail very difficult integrations in four dimensions, and exemplifying this by our rational results (45, 51, 52), which give the two-loop renormalization constants (44, 49, 50).

5.2 Laurent expansion for one-lepton QED

Before giving the $\omega \rightarrow 0$ behaviour of (44, 49, 50), there is an important observation to make regarding the D -dimensional physical charge e_R , lest our subsequent formulae be misunderstood.

In D dimensions, the on-shell charge, e_R , necessarily has mass dimension $(4-D)/2 \equiv \omega$. This is an ineluctable consequence of having a dimensionless action [31]. It follows that the L -loop term of the expansion of a dimensionless quantity (such as $g-2$ or a renormalization constant) will involve e_R^{2L} divided by some physical mass or momentum scale (such as M) to the power $2L\omega$, as is the case in (44, 49, 50). There will also be the inevitable factor of $(4\pi/e^\gamma)^{L\omega}$ which results from the surface of the unit sphere in D -dimensions, $2\pi^{D/2}/\Gamma(D/2)$, divided by the $(2\pi)^D$ factor of Fourier transformation. It is therefore very convenient, though not logically necessary, to introduce the shorthand notation

$$\alpha_M \equiv \frac{e_R^2}{4\pi} \left(\frac{4\pi}{M^2 e^\gamma} \right)^\omega \quad (\text{not a running coupling}). \quad (53)$$

The important point is that when one has obtained a result for a finite quantity, such as $g-2$, one may take the limit $\omega \rightarrow 0$ and express the answer in terms of the experimentally determined 4-dimensional coupling

$$\alpha \equiv \lim_{D \rightarrow 4} \alpha_M = 1/137.036 \dots \quad (\text{for all } M). \quad (54)$$

By this device we are able to present two-loop results uncluttered by factors from the expansion

$$\left(\frac{\alpha_M}{\pi\omega} \right)^2 = \left(\frac{e_R^2}{4\pi^2} \right)^2 \{ \omega^{-2} + 2(\log 4\pi - \gamma - \log M^2)\omega^{-1} + 2(\log 4\pi - \gamma - \log M^2)^2 + O(\omega) \}$$

which, whilst formally correct, looks dimensionally puzzling at first sight. What it means is that one should use the same mass unit to express the values both of M and of e_R , for $\omega \neq 0$. Thus one might as well work with units in which $M^2 = 4\pi/e^\gamma$. Only when there is another mass scale in the problem, as in the next Sect., need one concern oneself with logarithms.

In terms of α_M , we find

$$Z_3 = 1 - \left\{ \frac{1}{3}\omega^{-1} + \frac{1}{6}\zeta(2)\omega + O(\omega^2) \right\} \frac{\alpha_M}{\pi} - \left\{ \frac{1}{8}\omega^{-1} + \frac{1}{6}\zeta(2)\omega + O(\omega) \right\} \frac{\alpha_M^2}{\pi^2} + O(\alpha_M^3) \quad (55)$$

$$Z_2 = 1 - \left\{ \frac{3}{4}\omega^{-1} + 1 + \left(\frac{3}{8}\zeta(2) + 2 \right)\omega + O(\omega^2) \right\} \frac{\alpha_M}{\pi} + \left\{ \frac{9}{32}\omega^{-2} + \frac{55}{96}\omega^{-1} + \pi^2 \log 2 - \frac{3}{2}\zeta(3) - \frac{211}{32}\zeta(2) + \frac{7685}{1152} + O(\omega) \right\} \frac{\alpha_M^2}{\pi^2} + O(\alpha_M^3) \quad (56)$$

$$Z_m = 1 - \left\{ \frac{3}{4}\omega^{-1} + 1 + \left(\frac{3}{8}\zeta(2) + 2 \right)\omega + O(\omega^2) \right\} \frac{\alpha_M}{\pi} + \left\{ \frac{5}{32}\omega^{-2} + \frac{155}{192}\omega^{-1} + \frac{1}{2}\pi^2 \log 2 - \frac{3}{4}\zeta(3) - \frac{87}{32}\zeta(2) + \frac{1169}{384} + O(\omega) \right\} \frac{\alpha_M^2}{\pi^2} + O(\alpha_M^3) \quad (57)$$

where, as ever in on-shell two-loop renormalization, one should retain the one-loop $O(\omega)$ terms, since they may later be multiplied by the one-loop $O(1/\omega)$ terms of another expansion. The numerical values of the finite parts of the coefficients of α_M^2/π^2 in (56) and (57) are 0.86 and 1.09, respectively, indicating considerable cancellations between the four terms in each analytical result.

5.3 Laurent expansion for multi-lepton QED

To two loops, the effect of adding more leptons is easy to specify in the case of Z_3 : given a set of leptons of masses $\{M_i | i=1, N_{lep}\}$, one merely replaces α_M in the one-loop term of (55) by $\sum_i \alpha_{M_i}$, and α_M^2 in the two-loop term by $\sum_i \alpha_{M_i}^2$. There are no cross terms, to two loops.

At first sight this might seem odd, since the bare self energy is iterated in $Z_3 = 1/(1 + \Pi(0))$, which does produce cross terms in the expansion in powers of the bare charge. However, these are removed when one performs one-loop charge renormalization. The corresponding effect on (46) is to replace $\log M/\mu$ by $\sum_i \log M_i/\mu$. Thus

the effect of the μ and τ leptons on the \overline{MS} coupling at the electron mass is rather substantial:

$$\frac{\pi}{\bar{\alpha}(M_e)} = \frac{\pi}{\alpha} + \frac{2}{3} \log \frac{M_\mu M_\tau}{M_e^2} + \frac{\alpha}{\pi} \left(\frac{1}{2} \log \frac{M_\mu M_\tau}{M_e^2} - \frac{15}{16} \right) + O(\alpha^2) + O(\omega). \quad (58)$$

Only in one-lepton QED is it a good approximation [31] to take $\bar{\alpha}(M_e) \approx \alpha$.

The changes to the renormalization constants (56, 57) of one lepton, with mass M , due to another lepton, with mass $M_i = rM$, are to add the following corrections

$$\Delta \bar{Z}_2 = \left\{ \frac{1}{16\omega} - \frac{1}{4} \log r - \frac{5}{96} + 2\bar{\Delta}(r) \right\} \frac{\alpha_M^2}{\pi^2} \quad (59)$$

$$\Delta \bar{Z}_m = \left\{ -\frac{1}{8\omega^2} + \frac{5+24 \log r}{48\omega} - \frac{1}{2} \log^2 r + \frac{2}{3} \log r + \frac{3}{8} \zeta(2) + \frac{71}{96} - 2\Delta(r) \right\} \frac{\alpha_M^2}{\pi^2} \quad (60)$$

where, given the gross disparity between lepton masses, it is a good approximation to work with the appropriate limiting forms of the dilogarithms (23, 27), given by (24, 28) when $r \gg 1$, or by (25, 29) when $r \ll 1$.

Note that on-shell charge renormalization ensures that the mass-dependent singular term, $\omega^{-1} \log r$, in the bare correction (22), is absent from the renormalized correction (59). Correspondingly, the absence of such a term from (26) entails its appearance in (60). There seems, in general, to be no particular reason why either renormalization constant should be well-behaved as $r \rightarrow \infty$, since only relationships between observable quantities satisfy decoupling theorems. The mass singularities of renormalization will cancel those in the truncated bare Green functions, to ensure decoupling of internal heavy-lepton effects from renormalized light-lepton Green functions.

6 Summary and conclusions

In dimensionally regularized QED, Z_2 is gauge-invariant to all orders, by virtue of the Johnson-Zumino [12] identity (6). We are not aware of a non-abelian generalization of this result. Nevertheless, Z_2 is gauge-invariant at the two-loop level in QCD, thanks to intricate cancellations between the diagrams of Fig. 1. We take this as strong evidence of its gauge invariance in general.

The precise form of Z_2 at two loops provides a link between the MS renormalization of a heavy-quark field in QCD and in the effective field theory [13, 14] obtained by letting $M \rightarrow \infty$. To convert MS-renormalized truncated Green functions to on-shell S-matrix elements in QCD and EFT one must multiply by the factors $(Z_2^{\text{MS}}(\mu)/Z_2)^{N_E/2}$ and $(\tilde{Z}_2^{\text{MS}}(\mu)/\tilde{Z}_2)^{N_E/2}$, respectively, for processes with N_E external heavy quarks. But in dimensionally regularized EFT, with one or more infinite-mass quarks and N_L zero-mass quarks, there is *no* on-shell wave-function renormalization, since the on-shell self energy is scale free. Thus ratios of S-matrix elements differ from ratios of renormalized Green functions by powers of the factor $R(\mu) \equiv (Z_2^{\text{MS}}(\mu)/Z_2)/\tilde{Z}_2^{\text{MS}}(\mu)$, where Z_2^L includes the effects of light quarks and gluons in QCD, but excludes the effects of heavy-quark loops, since these are discarded in EFT. The factor $R(\mu)$ must be finite. Its μ dependence is therefore determined by the singular terms in Z_2^L , from which we have obtained the gauge-invariant difference (35) between the anomalous dimensions of the heavy-quark field in QCD and EFT. Renormalization-group improvement then gives

$$R(\mu) \approx R(M) \left(\frac{\tilde{\alpha}_s(M)}{\tilde{\alpha}_s(\mu)} \right)^{4/b_1} \left(\frac{1 + E_2 \tilde{\alpha}_s(M)/\pi}{1 + E_2 \tilde{\alpha}_s(\mu)/\pi} \right)$$

where $b_1 = 11 - \frac{2}{3}N_L$, $E_2 = \frac{175}{162}$ or $\frac{4233}{3750}$ for $N_L = 3$ or 4, and the integration constant is found from the finite part of Z_2^L to be

$$R(M) \approx 1 + \frac{4}{3} \tilde{\alpha}_s(M)/\pi + (19.23 - 1.33 N_L) \tilde{\alpha}_s^2(M)/\pi^2$$

whose two-loop term is commensurate with three-loop MS renormalization and, like the corresponding term [7] in $Z_m^{\text{MS}}(M)/Z_m$, is numerically large.

These results were obtained from the exact D -dimensional rational functions of Tables 1 and 2, found by implementation of the recurrence relations of [7] in a REDUCE [23] program which involves no integration whatsoever. For convenience, the resultant Laurent expansions are given in Table 3, before coupling renormalization, and Table 4, after MS renormalization of the QCD coupling. The EFT anomalous dimension

$$\tilde{\gamma}_F = \frac{(\bar{a}-3) C_F \tilde{\alpha}_s}{2\pi} + \left\{ \left(\frac{\bar{a}^2}{32} + \frac{\bar{a}}{4} - \frac{179}{96} \right) C_A + \frac{2}{3} T_F N_L \right\} \frac{C_F \tilde{\alpha}_s^2}{\pi^2} + O(\tilde{\alpha}_s^3)$$

was obtained from the singular terms of Table 4 and the corresponding QCD result [20]. It has been verified [16] by an analogous implementation of the recurrence relations for the off-shell two-loop integrals of EFT.

This complete avoidance of integration, or infinite summation, is familiar in massless QCD [22] and clearly capable of extension to EFT. What is more surprising is that the two-loop on-shell two- and three-point functions of pure QED fall into the same category of rational simplicity in D dimensions, as exemplified by the complete account of two-loop renormalization given, for all D , by (44, 49, 50) and, for $D \rightarrow 4$, by (55–57). The classic two-loop result for $g-2$ may also be viewed as a calculation of the $D \rightarrow 4$ limits of the three coefficients of the integral structures (14) to which all on-shell two-loop diagrams of the type of Fig. 1 are systematically reducible. Indeed the value [18]

$$g-2 = \alpha/\pi + (\zeta(2) - I(0) + \frac{197}{2}) \alpha^2/\pi^2 + O(\alpha^3)$$

clearly demonstrates that $I(0) = \pi^2 \log 2 - \frac{3}{2} \zeta(3)$ is central to on-shell two-loop QED. This $D=4$ value of the integral (15) was obtained in [17] by evaluation of trilogarithmic integrals. But even that is unnecessary, since recently it has proved possible [19] to expand $I(\omega)$ through $O(\omega^2)$ by purely algebraic methods. This expansion involves a fifth-order polylogarithm, $\text{Li}_5(\frac{1}{2})$

$$= \sum_{n=1}^{\infty} 2^{-n} n^{-5}, \text{ typical of four-loop QED calculations, yet}$$

no integration is needed to obtain it.

When one encounters a physically significant mass ratio, such as M_μ/M_μ in the calculation of the muon's anomalous magnetic moment, exact two-loop calculation entails the evaluation of dilogarithms, by old-fashioned analytical techniques. We have given the corresponding effects (59, 60) on renormalization constants in terms of the dilogarithms (23, 27), whose limiting forms (24, 28) and (25, 29) are useful in QED.

In conclusion: on-shell renormalization of a theory with a single mass scale enjoys much of the calculational simplicity of deep-euclidean MS renormalization. Its results, however, are more powerful, since they determine both the MS counterterms and the finite parts needed to make contact with physical processes. On-shell renormalization is also satisfyingly gauge-invariant. The physical significance of this is that the gauge dependences

of MS renormalization of QCD and EFT cancel. The implications of our results for the two-loop anomalous dimensions of EFT currents [14, 15] linking static and massless quarks are under study [16], as are the prospects of extending our methods for massive Feynman integrals to three loops [19].

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Appendix C

REDUCE programs

The following REDUCE programs are reproduced here.

`feyn.rd3` These are the Feynman rules, as in appendix D.2, in REDUCE form. [page 101]

`minnie.rd3` This is the program which implements the recurrence relations discussed in section 3.3, namely equations (3.19) and (3.20), and equations (3.23) and (3.24). [page 102]

`residue.rd3` This program produces Laurent expansions of the series Z_2 , the wavefunction renormalisation constant, and $Z_M = m_0/M$, the ratio of the pole to the bare mass. It is referred to in section 4.3. [page 110]

`dpk4.rd3` This program calculates the recurrence relations of tables 6 and 7. For discussion, see page 49. [page 119]

C.1 Feynman rules

These are the Feynman rules used in the suite of recurrence relation programs. Note that there are no colour factors present.

```

PROCEDURE QP (l,p)$                                [ Quark propagator
i * (g(l,p) + m0) / (p.p - m0^2)$

PROCEDURE QPO (l,p)$                                [ Light quark propagator
i * g (l,p) / p.p$

PROCEDURE GP (u,v,k)$                                [ Gluon propagator
-i * (u.v + (a0-1) * k.u * k.v / k.k) / k.k$      [ a0 = a0, the gauge parameter

PROCEDURE HP (p)$                                    [ Ghost propagator
i / p.p$

PROCEDURE HHGV (al,p)$                                [ Ghost-ghost-gluon vertex
-g0 * p.al$

PROCEDURE G3V (al,p,be,q,ga,r)$                      [ Three-gluon vertex
-g0 * (be.ga * (q-r).al                               [ g0 = g0, the coupling constant
+ ga.al * (r-p).be
+ al.be * (p-q).ga )$

PROCEDURE GQQV (l,al)$                                [ Gluon-quark-quark vertex
i * g0 * g (l,al)$

$end$

```

C.2 minnie.rd3

This is a general version of the MINNIE program. Rather than write an incrementally different version for each use of these routines, this program will do its stuff on a general array, MINNIE, of length MINNIELEN, which are set up before the program is invoked.

The program produces no output of its own, other than the diagnostic messages produced by the procedure ANOMALY, and diagnostic counters within for...do loops, which are enclosed in COMMENT...\$

It is worth mentioning that the invariants in this program are *Euclidean* invariants, rather than the Minkowski invariants described in the text.

```
write "comment"$
```

```
factor m5,n5$
```

```
load solve $ [ explicitly loads solve module
```

```
vector p,k,l,q,r$
```

```
vecdim d$
```

```
d := 4 - 2*w$
```

These should go here, rather than down with the calculation of the HARDs, so that d is defined in terms of ω before any of the II() etc are evaluated—otherwise, lots of gammawierd()'s appear in the output.

```
operator nbg,nde,nad,n5hard,hard,hard1,hard2,anomalous$
```

```
operator ii,gg,fabc$
```

```
factor ii,gg,fabc,hard,hard1,hard2,anomalous$
```

```
symmetric hard,gg$
```

```
anomalycounter := 0$
```

```
procedure anomaly (sect,fname,p1,p2,p3,p4,p5)$
```

```
<<
```

```
anomalycounter := anomalycounter + 1$
```

```
write "Anomalous function nr.(",anomalycounter,") in section ",sect,
```

```
    " is ",fname,"(",p1,",",p2,",",p3,",",p4,",",p5,")"$
```

```
anomalous (anomalycounter)
```

```
>>$
```

```
FOR ALL NA,NC,ND,NE,NG SUCH THAT NG<1 LET
```

```
    N5 (NA,NC,ND,NE,NG) = NBG (NA,0,NC,ND,NE,NG)$ [ And thence to M5()
```

```
FOR ALL NA,NC,ND,NE,NG SUCH THAT NG>0 AND MIN(ND,NE)<1 LET
```

```
    N5 (NA,NC,ND,NE,NG) = NDE (NA,NC,ND,NE,NG)$ [ And thence to M5() via NAD
```

```
FOR ALL NA,NC,ND,NE,NG SUCH THAT MIN(ND,NE,NG)>0 LET
```

```
    N5 (NA,NC,ND,NE,NG) = N5HARD (NA,NC,ND,NE,NG)$ [ And thence to HARD(), HARD1()
```

```

procedure nbg (na,nb,nc,nd,ne,ndd)$
if ndd=0 then
    m5 (na,nb,nc,nd,ne)
else
    +nbg (na-1,nb ,nc ,nd ,ne ,ndd+1)
    -nbg (na ,nb-1,nc ,nd ,ne ,ndd+1)
    +nbg (na ,nb ,nc-1,nd ,ne ,ndd+1)
    +nbg (na ,nb ,nc ,nd-1,ne ,ndd+1)
    +nbg (na ,nb ,nc ,nd ,ne-1,ndd+1)$

```

```

procedure nde (na,nc,nd,ne,ng)$
if ne<1 and (ne>nd or nd>0) then
    nde (nc,na,ne,nd,ng)
else
    nad (0,na,nc,ng,ne,nd)$

```

[Has nd or ne ≤ 0

[Ends up with nd ≤ 0 and (ne > 0 or
ne $> nd$)

```

procedure nad (na,nb,nc,nd,ne,ndd)$
if ndd=0 then
    m5 (na,nb,nc,nd,ne)
else
    -nad (na-1,nb ,nc ,nd ,ne ,ndd+1)
    +nad (na ,nb-1,nc ,nd ,ne ,ndd+1)
    +nad (na ,nb ,nc-1,nd ,ne ,ndd+1)
    +nad (na ,nb ,nc ,nd-1,ne ,ndd+1)
    -nad (na ,nb ,nc ,nd ,ne-1,ndd+1)$

```

```

procedure n5hard (na,nc,nd,ne,ng)$
if min(nd,ne)<1 then
    nde (na,nc,nd,ne,ng)
else if max(na,nc)<0 then
    anomaly (1,"n5hard",na,nc,nd,ne,ng)
else if nc<na then
    n5hard (nc,na,ne,nd,ng)
else if nc=0 then <<
    if na=0 then
        hard (nd,ne,ng)
    else if na=-1 then
        hard1 (nd,ne,ng)
    else if na=-2 then
        hard2 (nd,ne,ng)
    else
        anomaly (2,"n5hard",na,nc,nd,ne,ng) >>
else
    (ne * n5hard (na ,nc-1,nd ,ne+1,ng )
    +ng *(n5hard (na ,nc-1,nd ,ne ,ng+1)
    -n5hard (na ,nc ,nd-1,ne ,ng+1)))

```

[Because of last case

[$\rightarrow M5()$

[Procedure wouldn't terminate otherwise

[Before if nc=0

```
/(d - 2*nc - ne - ng)$
```

```
for all a,b,c such that a>b let
  fabc (a,b,c) = fabc (b,a,c),
  hard1 (c,a,b) = hard1 (c,b,a)$
for all a,b let
  hard (a,b,0) = ii(0,a) * ii(0,b)$
```

[Tidy up the indices of diagrams 1 & 2

```
operator mb,mde$
factor mb,mde$
```

[These will be made into procedures

Reduce the M5's to II's, GG's and FABC's.

This must be done by defining a procedure M5PROC and then using a LET statement to set m5 equal to M5PROC. If the procedure were simply called M5, then any M5's present before the definition would be ignored.

```
procedure m5proc (na,nb,nc,nd,ne)$
if min(nd,ne)<1 then
  mde (na,nb,nc,nd,ne)
else
  if nb<1 then
    mb (na,nb,nc,nd,ne)
  else
    if (na = 0 and nc = 0) then
      fabc (nd,ne,nb)
    else if min(na,nc)>1 then
      anomaly (4,"m5",na,nb,nc,nd,ne)
    else if na>0 and nc<=0 then
      m5proc (nc,nb,na,ne,nd)
    else
      (na * (m5proc (na+1,nb-1,nc ,nd ,ne )
        -m5proc (na+1,nb ,nc-1,nd ,ne ))
      +nd * (m5proc (na ,nb-1,nc ,nd+1,ne )
        -m5proc (na ,nb ,nc ,nd+1,ne-1)))
      /(d - 2*nb - na - nd) $
```

[NB,ND,NE > 0

[This is redundant, but it's nice to keep it here

[[1]

[[2] ...

[[3]

This is all a bit tricky.... The procedure will terminate obviously when $n_d \leq 0$ or $n_e \leq 0$, when $n_b \leq 0$, or when $n_a = n_c = 0$. In the path marked [2], three of the four recursive invocations of m5proc reduce n_b or n_e , and thus drive the thing toward termination. M5proc will not be reinvoked in line [3] if $n_a = 0$ —this branch will therefore terminate if $n_a \leq 0$. Line [1], and the $nc-1$ in line [3] guarantee that this will be so eventually.

```
for all na,nb,nc,nd,ne let
  m5 (na,nb,nc,nd,ne) = m5proc (na,nb,nc,nd,ne)$
```

```

FOR N := 1:MINNIELEN DO <<
  write n $
  MINNIE (N) := MINNIE (N)$ >> $

```

With that done, and the various mb's and mde's collected via the factor declaration, we can now define the procedures MBPROC() and MDEPROC() to evaluate the operators MB() and MDE().

This has to be done in the same way as above.

```

procedure mdeproc (na,nb,nc,nd,ne)$
                                                                    [  $n_d \leq 0$  or  $n_e \leq 0$  ]
if nd>0 then
  mdeproc (nc,nb,na,ne,nd)
                                                                    [  $n_e \leq 0$  ]
else
  if nd<0 then
    mdeproc (na-1,nb ,nc ,nd+1,ne )
    +m0^2 * ( 2*(nd+1) * mdeproc (na-1,nb ,nc ,nd+2,ne )
              + ne      *(mdeproc (na-1,nb ,nc ,nd+1,ne+1)
                          -mdeproc (na ,nb-1,nc ,nd+1,ne+1)
                          +mdeproc (na ,nb ,nc-1,nd+1,ne+1)))
    /(3*d/2 - na - nb - nc - nd - ne - 1)
  else
                                                                    [  $n_d = 0$  ]
    if min(na,nb,ne)<1 then
      0
                                                                    [ from properties of II() and GG() ]
    else
      gg (na,nb) * ii (na+nb+nc-d/2, ne)$

procedure mbproc (na,nb,nc,nd,ne)$
                                                                    [  $n_{d,e} > 0$  and  $n_b \leq 0$  ]
if min(nd,ne)<1 then
  0
                                                                    [  $n_{d,e}$  are lowered below ]
else
                                                                    [  $n_{d,e} > 0$  ]
  if nb=0 then
    ii (na,nd) * ii (nc,ne)
  else
                                                                    [  $n_b < 0$  ]
    if ne>1 then
      ( mbproc (na-1,nb+1,nc ,nd+1,ne-1) * 2*nd
        +(mbproc (na-1,nb+1,nc ,nd ,ne-1)
          -mbproc (na ,nb+1,nc ,nd-1,ne-1)) * (3*d/2-na-nb-nc-nd-ne)/m0^2
        )/(ne-1)
      + mbproc (na-1,nb+1,nc ,nd ,ne )
      + mbproc (na ,nb+1,nc-1,nd ,ne )
    else if nd>1 then
      mbproc (nc,nb,na,ne,nd)
    else
                                                                    [  $n_d = n_e = 1$  ]
      (nd * mbproc (na-1,nb ,nc ,nd+1,ne )
       +ne * mbproc (na ,nb ,nc-1,nd ,ne+1)
       )/(2*(d-na-nb-nc) - nd - ne)$

```

```

for all na,nb,nc,nd,ne let
  mb (na,nb,nc,nd,ne) = mbproc (na,nb,nc,nd,ne),
  mde (na,nb,nc,nd,ne) = mdeproc (na,nb,nc,nd,ne)$

FOR N := 1:MINNIELEN DO <<
  write n $
  MINNIE (N) := MINNIE (N)$ >> $

out"reduce_out:just_hards.nat"$
off nat, echo $
procedure onefun (z)$ 1$

write "array justhards (",minnielen,")" $
for n := 1:minnielen do
  write "justhards(",n,") := ",
    sub (g0=1/2, a0=1/3, discrim=1/5, w=1/7, m0=1/11, gamma0=onefun,
      minnie(n)) $

write "$END$$"
on nat, echo $
shut"reduce_out:just_hards.nat"$

```

Now work out the values of hard and hard1. After this, the only remaining hard integral is the integral NHARD.

```
operator a,b,ab,f$
```

```

p.p := a(6)$
k.k := a(1)$
l.l := a(2)$
p.k := (a(3) - a(1))/2$
p.l := (a(4) - a(2))/2$
k.l := (a(5) - a(3) - a(4))/2$

```

[Use the following invariants:

$a(n)$ is the invariant, $-b(n)$ is differentiation of an integrand w.r.t. $a(n) \Rightarrow b(n) = -\frac{d}{da(n)}$.

```

for all m,n such that (m=1 or m=2) and not (n=1 or n=2)
  match a(m) * b(n) = ab (m,n)$
match a(6) * b(6) = ab(6,6)$

```

```

procedure integrand(q,r)$
2 * (
  if q=k then
    b(1)*k.r + b(3)*(p+k).r + b(5)*(p+k+1).r
  else if q=l then

```

```

      b(2)*l.r + b(4)*(p+1).r + b(5)*(p+k+1).r
else if q=p then
      b(3)*k.r + b(4)*l.r      + b(5)*(k+1).r      + b(6)*p.r
else
      anomaly (5,integrand,q,r,0,0,0)
) - (if r=q then d else 0)$

simults := { integrand (k,k),  integrand (k,l),  integrand (k,p),
              integrand (l,k),  integrand (l,l),  integrand (l,p),
              integrand (p,k),  integrand (p,l),  integrand (p,p)}$

```

Out of a total of 36 possible bilinears, these 9 are chosen as the ones to be eliminated. The choice is not entirely obvious.

```

freevars := {ab (1,3),  ab (1,4),  ab (1,5),  ab (1,6),
              ab (2,3),  ab (2,4),  ab (2,5),  ab (2,6),
              ab (6,6)}$

solutions := solve (simults, freevars)$

procedure righthandside(n)$
rhs(part(first(solutions),n))$

for n := 3:6 do <<
      ab (1,n) := righthandside (n-2)$
      ab (2,n) := righthandside (n+2)$>>$
ab (6,6) := righthandside (9)$

solvefailed := 0 $
for each x in simults do if x neq 0 then solvefailed:=solvefailed + 1 $
if solvefailed > 0 then <<
      write "Solve failed!" $
      write simults := simults $ >> $

noncom a,b,f,ab$

b(1) := b(2) := 0$

ab (1,3) := ab (1,3)$
ab (1,6) := ab (1,6)$

```

Now define the effect of the operators $a(n)$ and $b(n)$ on the 'proto-hard' function $f()$.

```

for all na,nb,nc let
      a(3) * f(na,nb,nc) = f(na-1,nb ,nc ),
      a(4) * f(na,nb,nc) = f(na ,nb-1,nc ),
      a(5) * f(na,nb,nc) = f(na ,nb ,nc-1),
      a(6) * f(na,nb,nc) = -m0^2 * f(na,nb,nc),
      b(3) * f(na,nb,nc) = na * f(na+1,nb ,nc ),
      b(4) * f(na,nb,nc) = nb * f(na ,nb+1,nc ),
      b(5) * f(na,nb,nc) = nc * f(na ,nb ,nc+1),

```

$$b(6) * f(na,nb,nc) = -(3/2*d-na-nb-nc) / m0^2 * f(na,nb,nc)$$$

```
method3 := sub (f=hard, ab(1,3)*f(naa-1,nbb,ncc)/(naa-1) ) $
method6 := sub (f=hard, ab(1,6)*f(naa,nbb,ncc)*m0^2/(naa+nbb+ncc-3/2*d) ) $

procedure generate(na,nb,nc) $
sub(naa=na,nbb=nb,ncc=nc, method3 - method6) $
```

Now solve the expressions generated by this difference for the seven `hard()`'s that appear.

```
simults := { generate(2,1,1),
               generate(2,2,1),
               generate(3,1,1),
               generate(2,2,2),
               generate(3,2,1),
               generate(4,1,1),
               n5(1,1,1,1,1) - m0^(2*d-10)*nhard } $

freevars := {hard(1,1,1), hard(1,1,2), hard(1,2,2), hard(1,1,3),
              hard(2,2,2), hard(1,2,3), hard(1,1,4)} $

solutions := solve (simults,freevars) $

hard (1,1,1) := righthandside (1) $
hard (1,1,2) := righthandside (2) $
hard (1,2,2) := righthandside (3) $
hard (1,1,3) := righthandside (4) $
hard (2,2,2) := righthandside (5) $
hard (1,2,3) := righthandside (6) $
hard (1,1,4) := righthandside (7) $

for all na,nb,nc such that nc<0 let
  hard (na,nb,nc) = nbg (0,0,0,na,nb,nc) $           [ solve generates these 'hards'
```

hard2: Based on [thesis] eqn (3.21), before setting $\partial/\partial a_{1,2} = 0$

```
procedure hard2(h3,h4,h5)$                                [ from ab(1,6) on hard1(h3,h4,h5)
-((h5+h4+2*h3-4+4*w-8)*hard1(h3,h4,h5)
-h4*hard1(h3,h4+1,h5-1)
-h5*hard1(h3,h4-1,h5+1)
-(-(3*d/2+1-h3-h4-h5)/m0^2)*hard1(h3-1,h4,h5)           [ [+1 from a(1)*b(1)=-1'
+h5*hard1(h3-1,h4,h5+1)
+h4*hard1(h3-1,h4+1,h5))
/(-(3*d/2+1-h3-h4-h5)/m0^2)
$
```

this replacement of method 6 for `hard1` should make no difference to `a(2,0)`, `b(2,1)`


```

procedure hard1(h3,h4,h5)$
-((h5+h4+2*h3+4*w-8)*hard(h3,h4,h5)
-h4*hard(h3,h4+1,h5-1)
-h5*hard(h3,h4-1,h5+1)
-(-(3*d/2-h3-h4-h5)/m0^2)*hard(h3-1,h4,h5)
+h5*hard(h3-1,h4,h5+1)
+h4*hard(h3-1,h4+1,h5))
/(-(3*d/2-h3-h4-h5)/m0^2)
$

```

```

solvefailed := 0 $
for each x in simults do if x neq 0 then solvefailed:=solvefailed + 1 $
if solvefailed > 0 then <<
  write "Solve failed!" $
  write simults := simults $ >> $

```

```

dummy := elapsedtime$
array timings (minnielen)$

```

```

FOR N := 1:MINNIELEN DO <<
  write n $
  MINNIE (N) := MINNIE (N)$
  timings (n) := elapsedtime $ >> $

for n := 1:minnielen do write "time ",n," = ",timings (n) $

write "$"$

```

```

$END$

```

C.3 residue.rd3

This does the calculations for workings (47) and (47a), hopefully getting the numbers for Z_F in

$$S_F = \frac{Z_F}{p - M} + \text{regular.}$$

The thing calculates in terms of $\bar{\Sigma} = \frac{1}{4} \text{Tr} i(1 + \delta p)(-i\Sigma)$. To get $A_1 \dots B'_2$, we need to get $\sigma = d\bar{\Sigma}/d \ln p^2$, and then fiddle around extracting the juicy bits.

This routine inputs `minnie.nat`, which expresses `sigma1(1..3)`, `sigma2(1..7)` and `sigma2p(1..7)` in terms of `gamma0()`. It then extracts $A_2 \dots B'_2$ as detailed in (47).

For the residue, we have

$$iZ_F^{-1} = (1 - B) \left(1 - \frac{\partial m}{\partial p} \right) \Big|_{p=M}$$

where

$$\frac{\partial m}{\partial p} = 2 \frac{\partial \ln m}{\partial \ln p^2} = 2 \frac{A' - B'}{1 + A - B} + 2 \frac{B'}{1 - B}$$

All these are evaluated at $p = M$. So expand $A(p = M)$ as

$$\begin{aligned} A(\ln p^2 = \ln M^2) &= \Omega_M A_1(p^2 = M^2) + \Omega_M^2 A_2(p^2 = M^2) \\ B(M^2) &= \Omega_M B_1 + \Omega_M^2 B_2 \\ A'(M^2) &= \Omega_M(-\omega A_1 + A'_1) + \Omega_M^2(-2\omega A_2 + A'_2) \\ B'(M^2) &= \Omega_M(-\omega B_1 + B'_1) + \Omega_M^2(-2\omega B_2 + B'_2) \end{aligned}$$

where all the terms on the RHS are evaluated at $p^2 = M^2$, $\Omega_M = \Omega(p^2 = M^2)$ and terms of $O(\Omega^3)$ have been neglected. Then Taylor expand the RHS's to get:

$$\begin{aligned} A_n(M^2) &= A_n(m_0^2) + L A'_n(m_0^2) + \frac{1}{2} L^2 A''_n(m_0^2) \\ A'_n(M^2) &= A'_n(m_0^2) + L A''_n(m_0^2) \end{aligned}$$

where

$$L = \ln \frac{M^2}{m_0^2} = 2\Omega_M A_1(m_0^2) + O(\Omega^2)$$

We also, incidentally, want to output the values of the coefficients C1 and C2 for QCD and QED.

The meanings of the variables used are as follows:

<code>sigma2(n)</code>	$\mapsto \Sigma_{2,n}$	Contribution from 2-loop diagram n
<code>sigma2p(n)</code>	$\mapsto d\Sigma_{2,n}/d \ln p^2$	
<code>sigma1(n)</code>	$\mapsto d^{n-1}\Sigma_1/d(\ln p^2)^{n-1}$	From 1-loop diagram
<code>S</code>	$\mapsto \bar{\Sigma} = \sum \Sigma_{2,n}$	See above
<code>Sp</code>	$\mapsto \sigma = d\bar{\Sigma}/d \ln p^2$	
<code>a1p</code>	$\mapsto dA(p^2 = m_0^2)/d \ln p^2$	likewise for <code>a1pp</code> etc
<code>lnP</code>	$\mapsto \ln p^2$	
<code>Omega</code>	$\mapsto \Omega = g_0^2/((4\pi)^{D/2} M^{2\omega})$	NB, no $\Gamma(\omega)$, and $\Omega = \Omega(M)$
<code>w</code>	$\mapsto \omega$	in $D = 4 - 2\omega$
<code>M</code>	$\mapsto m_p$ or M	pole mass
<code>m0</code>	$\mapsto m_0$	parameter mass
<code>Zf</code>	$\mapsto -iZ_F$	

This program also calculates and outputs the terms in $Z_m = m_0/M$ where

$$Z_m = 1 + (C_{10} + C_{11}/\omega)\Omega + (C_{20} + C_{21}/\omega + C_{22}/\omega^2)\Omega^2$$

and we want to calculate $cnmr = C_{nm}$ (renormalised). Note that this definition of Z_m is inverse to the definition which was made for the first paper.

```

load hacks, factor, ezgcd, bfloat $

chc      := setoutput ("reduce_out:ci.nat") $
chd      := setoutput ("reduce_out:z_pole.nat") $
chabcd   := setoutput ("reduce_out:abcdz.nat") $

on echo, rat $

factor Omega $
let Omega^3 = 0 $

procedure ooop (x) $                                     [ One Over One Plus x
1 - x + x^2 - x^3 $

AM := Omega * a1m + Omega^2 * a2m $                      [ These are all at  $p^2 = M^2$  (incl Omega
AMp := Omega * (a1mp - w*a1m) + Omega^2 (a2mp - 2*w*a2m) $
BM := Omega * b1m + Omega^2 * b2m $
BMp := Omega * (b1mp - w*b1m) + Omega^2 (b2mp - 2*w*b2m) $

a1m := a1 + 2*Omega*a1 * a1p $                           [ Taylor expansion— $a_1, a_{1p}$  at  $p^2 = m_0^2$ 
a1mp := a1p + 2*Omega*a1 * a1pp $
b1m := b1 + 2*Omega*a1 * b1p $
b1mp := b1p + 2*Omega*a1 * b1pp $

a2m := a2 $ a2mp := a2p $ b2m := b2 $ b2mp := b2p $      [ +  $O(\Omega)$ 

C2 is the coefficient of  $\Omega_M^2$  in the expansion of  $Z_M = m_0/M$ . D1 (D2) is the unrenormalized coefficient
of  $\Omega_M$  ( $\Omega_M^2$ ) in the expansion of  $Z_F$ 

Might want to display results as functions of  $A_1 = A_1(m_0^2/p^2)$  etc. (47a.5) shows that  $A_1(p^2) =$ 
 $A_1(m_0^2/p^2)$ ,  $A_1'(p^2) = -A_1'(m_0^2/p^2)$  and  $A_1''(p^2) = A_1''(m_0^2/p^2) + A_1'(m_0^2/p^2)$ .

dlndlnp := (AMp-BMp) * ooop(AM-BM) + BMp * ooop(-BM) $    [  $\partial \ln m/\partial \ln p^2$ 
Zf := (1-BM) * (1-2*dlndlnp) $                             [ This is actually  $1/Z_f$ 
Zf := ooop (Zf-1) $                                         [ ... here we are
C1 := - a1 $
C2 := - a2 - a1 * (b1 + 2*a1p - a1) $                      [  $C_2, D_1, D_2$  in terms of  $a_1 \dots b_{2p}$ 
D1 := sub (omega=0, df (Zf,omega)) $
D2 := df (Zf,omega,2)/2 $

off echo,nat $
chtemp := switchoutput (chabcd) $

```

```

write c1 := c1 $ write c2 := c2 $
write d1 := d1 $ write d2 := d2 $
switchoutput (chtemp) $

```

```
clear Zf $
```

```
operator sigma1 $
```

```
IN"reduce_out:minnie.nat" $
```

```
[ defines sigma1(n), sigma2(n), sigma2p(n)
```

```
d := 4 - 2*w $
```

```
[  $T_F = 1/2$  at end and  $N_i = N_f - 1$ 
```

```
operator ncf, rcf $
```

The expansion coefficients $ncf = N_i$ and $rcf = R_j$ are

$$N_1 = C_A C_F, \quad N_2 = C_F^2, \quad N_3 = C_F N_F, \quad N_4 = C_F$$

$$R_1 = \Gamma^2(\omega), \quad R_2 = \frac{\omega \Gamma^2(-\omega) \Gamma(-4\omega) \Gamma(2\omega) \Gamma(\omega)}{\Gamma(-2\omega) \Gamma(-3\omega)}, \quad R_3 = I(\omega)$$

```
Colours := {
```

```
ncf(2) ~ ncf(1)/2,
```

```
[  $(C_F - \frac{1}{2}C_A)C_F$ 
```

```
-2 * ncf(4),
```

```
[  $-4T_F C_F$ 
```

```
ncf(2),
```

```
[  $C_F^2$ 
```

```
-1/2 * ncf(1),
```

```
[  $-\frac{1}{2}C_A C_F$ 
```

```
i/2 * ncf(1),
```

```
[  $\frac{1}{2}C_A C_F$ 
```

```
(-1)^2 * ncf(1),
```

```
[  $(-1)^2 C_A C_F$ 
```

```
-2 * (ncf(3) - ncf(4))}]$
```

```
[  $-4T_F N_i C_F = -4T_F (N_f - 1) C_F$ 
```

```
cf^2 := ncf(2)$
```

```
[ Appears when the one-loop term feeds  
into  $\Omega^2$  terms
```

```
let gamma0(w)^2 = rcf (1),
```

```
gamma0(-4w) = 1/(w * gamma0(-w)^2 * gamma0(2w) * gamma0(w))
```

```
* gamma0(-2w) * gamma0(-3w) * rcf (2),
```

```
nhard = rcf (3) $
```

```
factor ncf, rcf, gamma0 $
```

```
if numberp (fromfile) then
```

```
<<
```

```
write "Reading from residue.nat" $
```

```
in "reduce_out:residue.nat" $ >>
```

```
else <<
```

```
write "Generating, outputting to residue.nat" $
```

```
out"reduce_out:residue.nat"$
```

```
write "off echo" $
```

```
remark "S" $
```

```
let g0^2 = fourpi^(D/2) * m0^(2*w) $
```

This gets rid of g_0 , and various other nasty factors. Note that this does *not* mean that we are expanding Z_F about $\Omega = \Omega(m_0^2) = g_0^2 / ((4\pi)^{d/2} m_0^{2\omega})$ — we are extracting $A_n(m_0^2)$ from $\Sigma(p^2 = m_0^2)$ expanded in $\Omega(p^2)$ and so we *temporarily* need $\Omega = \Omega(m_0^2)$.

The coefficients of $\Omega(m_0^2)$ in $\Sigma(m_0^2)$ are calculated separately from each other in `quentin.rd3`.

```
FOR n := 1:3 DO SIGMA1(n) := SIGMA1(n) * i*Cf/fourpi^(d/2) /m0 $
S := FOR n := 1:7 SUM (SIGMA2(n) * part (colours, n) * (-1/fourpi^d) /m0) $
Sp := FOR n := 1:7 SUM (SIGMA2P(n) * part (colours, n) * (-1/fourpi^d) /m0) $
```

These multiplications arise as follows: the various colour factors are omitted from the original Feynman Rules, the factors of $i/(4\pi)^{d/2}$ come from the integrations, and the whole is divided by m_0 since it's the A_n and B_n we're interested in, and are about to extract.

```
clear g0^2 $

remark "1" $
write "write 1" $
write a1 := SUB (discrim=1/m0, Sigma1(1)) $
write b1 := a1 - SUB (discrim=0, Sigma1(1)) $

remark "1p" $
write "write 11" $
write alp := SUB (discrim=1/m0, Sigma1(2)) - b1 + w*a1 $
write b1p := alp - SUB (discrim=0, Sigma1(2)) - w*(a1-b1) $

remark "1pp" $
write "write 111" $
write alpp := SUB (discrim=1/m0, Sigma1(3)) + 2*w*alp - w^2*a1
              - 2*b1p + 2*w*b1 - b1 $
write b1pp := alpp - SUB (discrim=0, Sigma1(3)) - 2*w*alp + w^2*a1
              + 2*w*b1p - w^2*b1 $

remark "2" $
write "write 2" $
write a2 := SUB(discrim=1/m0, S) $
write b2 := a2 - SUB(discrim=0, S) $

remark "2p" $
write "write 22" $
write a2p := SUB(discrim=1/m0, Sp) - b2 + 2*w*a2 $
write b2p := - SUB(discrim=0, Sp) + a2p - 2*w*(a2-b2) $

write "$END$" $
shut"reduce_out:residue.nat"$
```

```
>> $
```

[End of if on generation/input of residue.nat

```
remark "Finished residue.nat" $
```

To convert to $A_n = A_n(m^2/p^2)$, we could do the following

```
a1pp := a1pp + a1p $    b1pp := b1pp
+ b1p $    a1p := -a1p $ b1p := -b1p $    a2p := -a2p $ b2p := -b2p $
```

```
chtemp := switchoutput (chabcd) $
write a1 := a1 $    write a1p := a1p$    write a1pp := a1pp $
write b1 := b1 $    write b1p := b1p$
switchoutput (chtemp) $
```

```
C1 := C1 $
C2 := C2 $
D1 := D1 $
D2 := D2 $
```

Form tables of coefficients of N_i and R_j in C_2 and D_2 . Having done that, re-express C_2 and D_2 as sums of terms in $1/\omega^2$, $1/\omega$ and regular.

```
array cij (4,3), dij (4,3), cdij (4,3) $
```

[cij holds C_2 , dij D_2 , cdij coefts of $D_2 + (1 + \frac{D}{4})C_2$]

```
for ii := 1:4 do for jj := 1:3 do <<
    cij (ii,jj) := df (C2, ncf(ii),rcf(jj)) $
    dij (ii,jj) := df (D2, ncf(ii),rcf(jj)) $ >> $
```

[These are in terms of ω]

```
operator gammahat$
factor gammahat$
for all z let gamma0(z) = gammahat(z)/z $
procedure gammahat(z)$
    1 + z2*z^2/2 - z3*z^3/3$
```

[$= \Gamma(1+z)$]

```
remfac ncf,rcf $
```

```
clear
```

```
gamma0(w)^2,
gamma0(-4w),
nhard $
```

```
let rcf(1) = gamma0(w)^2,
rcf(2) = w*gamma0(-w)^2*gamma0(-4w)*gamma0(2w)*gamma0(w)
    /gamma0(-2w)/gamma0(-3w),
rcf(3) = nhard $
```

```
factor ca,cf,nf $
let ncf(1) = ca * cf,
ncf(2) = cf^2,
ncf(3) = 2*tf*cf*nf,
```

```
ncf(4) = 2*tf*cf $
```

Write out the unrenormalised expressions for C_i and D_i .

```
remark "Writing out renormalised expressions" $
```

```
Zal1 := (11/12*ca - tf*nf/3) $
```

```
Zalm1 := - z2/6 $
```

```
C1 := C1 $
```

```
C2 := C2 $
```

```
D1 := D1 $
```

```
D2 := D2 $
```

[For some reason which I fail to fully understand, these two have to be here, rather than below, where they belong.

[For checking against on-shell QED.

[put in terms of ω

```
chtemp := switchoutput (chc) $
```

```
write "comment In terms of Colour factors and omega, unrenormalised" $
```

```
write c11 := sub (w=0, c1*w) $
```

```
    c1 := c1 - c11/w $
```

```
write c10 := sub (w=0, c1) $
```

```
    c1 := c1 - c10 $
```

```
write c1m1 := sub (w=0, c1/w) $
```

```
write c22 := sub (w=0, c2*w^2) $
```

```
    c2 := c2 - c22/w^2 $
```

```
write c21 := sub (w=0, c2*w) $
```

```
    c2 := c2 - c21/w $
```

```
write c20 := sub (w=0, c2) $
```

```
clear c1 $
```

```
clear c2 $
```

```
switchoutput (chd) $
```

```
write "comment In terms of Colour factors and omega, unrenormalised" $
```

```
write d11 := sub (w=0, d1*w) $
```

```
    d1 := d1 - d11/w $
```

```
write d10 := sub (w=0, d1) $
```

```
    d1 := d1 - d10 $
```

```
write d1m1 := sub (w=0, d1/w) $
```

```
write d22 := sub (w=0, d2*w^2) $
```

```
    d2 := d2 - d22/w^2 $
```

```
write d21 := sub (w=0, d2*w) $
```

```
    d2 := d2 - d21/w $
```

```
write d20 := sub (w=0, d2) $
```

```
clear d1 $
```

```
clear d2 $
```

Do renormalisation now.

Want to do coupling-constant renormalization, so we have to change $\Omega(p^2) = g_0^2 / ((4\pi)^{D/2} p^{2\omega})$ subject to the ren's'n

$$\frac{g_0^2}{4\pi} = \left(\frac{\mu^2 e^\gamma}{4\pi} \right)^\omega \alpha_s \left[1 - \frac{\alpha_s}{\pi} \frac{1}{\omega} \left\{ \frac{11}{12} C_A - \frac{T_f N_f}{3} \right\} \right].$$

Since we're only interested in the $1/\omega$, the $1/\omega^2$ and the terms regular at $\omega = 0$, we can set $\Omega = \frac{1}{4} \frac{\alpha_s}{\pi} [\dots]$. Here $\alpha_s = \alpha_s/\pi$. See (47a.5-6).

We can compare these results with the on-shell QED results in MZ2 if we include an $O(\alpha^2 w)$ term (Zalm1) in the coupling constant renormalisation.

remark "Doing renormalisation" \$

```
switchoutput (chc) $
on div $
write "comment renormalised" $
write c1m1r := 1/4*c1m1 $
write c10r := 1/4*c10 $
write c11r := 1/4*c11 $
write c20r := 1/16*c20 - 1/4*c1m1*zal1
      - 1/4*c11*Zalm1*Zalmiswitch $
write c21r := 1/16*c21 - 1/4*c10*zal1 $
write c22r := 1/16*c22 - 1/4*c11*zal1 $
```

[These expressions come from g_renorm.rd3
Zalmiswitch ∈ {0,1} turns this term on
and off

```
switchoutput (chd) $
write "comment renormalised" $
write d1m1r := 1/4*d1m1 $
write d10r := 1/4*d10 $
write d11r := 1/4*d11 $
write d20r := 1/16*d20 - 1/4*d1m1*zal1
      - 1/4*d11*Zalm1*Zalmiswitch $
write d21r := 1/16*d21 - 1/4*d10*zal1 $
write d22r := 1/16*d22 - 1/4*d11*zal1 $
```

[...and do exactly the same for D;

```
Zalmiswitch := 0 $
```

```
tf := 1/2 $
```

```
write "comment for QCD" $
write Z10r := d10r $
write Z11r := d11r $
write Z20r := d20r $
write Z21r := d21r $
write Z22r := d22r $
```

[This gives QCD

```
write "% Now QED!" $
```

```
tf := 1 $
```

```
cf := 1 $
```



```

ca := 0 $
nf := 1 $

switchoutput (chc) $
write "comment QED" $
temp := c1mr $      write "c1mr = ", temp $
temp := c10r $      write "c10r = ", temp $
temp := c11r $      write "c11r = ", temp $
temp := c20r $      write "c20r = ", temp $
temp := c21r $      write "c21r = ", temp $
temp := c22r $      write "c22r = ", temp $

switchoutput (chd) $
write "comment QED" $
temp := d1mr $      write "d1mr = ", temp $
temp := d10r $      write "d10r = ", temp $
temp := d11r $      write "d11r = ", temp $
temp := d20r $      write "d20r = ", temp $
temp := d21r $      write "d21r = ", temp $
temp := d22r $      write "d22r = ", temp $

Zalmiswitch := 1 $
[ check by comparing with on-shell results
  in MZ2

remark "Cross-checking with on-shell results" $

switchoutput (chc) $
write "comment QED - on-shell coupling renormalisation" $
temp := c1mr $      write "c1mr = ", temp $
temp := c10r $      write "c10r = ", temp $
temp := c11r $      write "c11r = ", temp $
temp := c20r $      write "c20r = ", temp $
temp := c21r $      write "c21r = ", temp $
temp := c22r $      write "c22r = ", temp $

switchoutput (chd) $
write "comment QED - on-shell coupling renormalisation" $
temp := d1mr $      write "d1mr = ", temp $
temp := d10r $      write "d10r = ", temp $
temp := d11r $      write "d11r = ", temp $
temp := d20r $      write "d20r = ", temp $
temp := d21r $      write "d21r = ", temp $
temp := d22r $      write "d22r = ", temp $

```

Now put in numerical values—return to MS coupling renormalisation

Zalmiswitch := 0 \$

remark "Numerical values" \$

on bigfloat, numval \$

z2 := $\pi^2/6$ \$

z3 := 1.2020569032 \$

nhard := $\pi^2 * \log(2) - 3/2 * z3$ \$

switchoutput (chc) \$

write "comment QED - numval" \$

write cim1r:= cim1r \$

write c10r := c10r \$

write c11r := c11r \$

write c20r := c20r \$

write c21r := c21r \$

write c22r := c22r \$

write "end" \$

switchoutput (chd) \$

write "comment QED - numval" \$

write dim1r:= dim1r \$

write d10r := d10r \$

write d11r := d11r \$

write d20r := d20r \$

write d21r := d21r \$

write d22r := d22r \$

write "end" \$

off float, numval \$

clear d \$

w := (4-d)/2 \$

off div \$

on factor, gcd, ezgcd, nat \$

for ii := 1:4 do for jj := 1:3 do

 write cdij (ii,jj) := dij (ii,jj) - (1+d/4)*cij(ii,jj) \$

clear w \$

shutoutput (chd) \$

shutoutput (chc) \$

shutoutput (chabcd) \$

\$end\$

C.4 Recurrence relations

Follows W(60).

Central to this is the function $ff(\alpha_i) = \prod^6 a_i^{-\alpha_i}$, where the a_i are the *Minkowski* invariants $a_1 = k^2$ to $a_6 = p^2 = m_0^2$, and $\alpha_6 \equiv 3D/2 - \sum^5 \alpha_i$. This is distinct from the Euclidean invariants in some other programs. The operators *downa* and *upa* operate on this by lowering and raising the arguments of *ff*. They are defined by $upa(n) \equiv \partial/\partial a_n$ and $downa(n) \equiv a_n$. They do not commute, thus

$$\begin{aligned} downa(n) \ upa(n)f &= -\alpha_n f \\ upa(n) \ downa(n)f &= -(\alpha_n - 1)f. \end{aligned}$$

The bilinears $ab(n,m)$ are defined as

$$\begin{aligned} ab(n,m) &\equiv -a_n \frac{\partial}{\partial a_m} = -downa(n) \ upa(m) \\ &= \alpha_m N^- M^+ \end{aligned}$$

When translating the *ab*'s into raising and lowering operators, there is some subtlety surrounding the powers of p^2 :

$$ab(6,5)f(\alpha_i) \equiv -p^2 \frac{\partial}{\partial a_5} f(\alpha_i) = +\alpha_5 f(\alpha_5 + 1)$$

where the p^2 has disappeared because of the *definition* of $\alpha_6 \equiv 3D/2 - \Sigma$, and in

$$ab(3,6)f(\alpha_i) \equiv \alpha_3 \frac{\partial}{\partial p^2} f(\alpha_i) = -\alpha_6 f(\alpha_3 - 1)$$

the powers work out for the same reason.

$$\begin{aligned} dk &\mapsto \partial/\partial k^\mu \\ dl &\mapsto \partial/\partial l^\mu \\ dp &\mapsto \partial/\partial p^\mu \\ dp2 &\mapsto \partial/\partial p^2 \\ ku &\mapsto k^\mu \\ lu &\mapsto l^\mu \\ pu &\mapsto p^\mu \\ pk &\mapsto k^2 + 2p \cdot k \\ pl &\mapsto l^2 + 2p \cdot l \\ klm &\mapsto (k-l)^2 \\ klp &\mapsto (k+l)^2 + 2p \cdot (k+l) \end{aligned}$$

off raise\$

OPERATOR dk, dl, dp, dp2, nn, mm \$
FACTOR nn, mm \$

OPERATOR

upa, downa,
ff,
upaf,

[raise and lower arguments of...

[like the function *N*

[equivalent to $upa(n)*ff(a)$ —fixes the *upa*
to the *ff*

```

ab, aa $
NONCOM upa, downa, ff, ab, upaf $

```

[ab is the bilinear, aa(n) is ab(n,n)

Note that, in order for the NONCOM declaration to work properly, ff must *always* be used with an argument, even though the argument has no meaning in this routine. ff.

```

LOAD hacks $

```

```

chops := SETOUTPUT ("dpk4_ops.nat") $

```

```

FOR ALL x,y LET dk (x*y) = (dk(x)) * y + x * (dk(y)),
                  dl (x*y) = (dl(x)) * y + x * (dl(y)),
                  dp (x*y) = (dp(x)) * y + x * (dp(y)),
                  dk (x/y) = (y* dk x - x *dk y)/y^2,
                  dl (x/y) = (y* dl x - x *dl y)/y^2,
                  dk (x+y) = dk x + dk y,
                  dl (x+y) = dl x + dl y $

```

[Differentiation rules
[product rule
[quotient rule
[linearity

```

FOR ALL n,x SUCH THAT FIXP (n) LET
    dk (n*x) = n * dk x,
    dl (n*x) = n * dl x,
    dk (-x)  = - dk x,
    dl (-x)  = - dl x $

```

[linearity, again

We would like to arrange that ku, lu, pu didn't commute with the operators. Reduce, however, doesn't seem to want to let us do this. Since the three vectors are always *in front* of the operators, however, we can fake this by telling Reduce to internally order them first.

This works!!!

```

KORDER ku, lu, pu $

```

```

LET ku * ku = downa(1),
    lu * lu = downa(2),
    pu * pu = downa(6),
    ku * pu = (downa(3) - downa(1))/2,
    lu * pu = (downa(4) - downa(2))/2,
    ku * lu = (downa(5) - downa(3) - downa(4))/2 $

```

```

LET dk pu = 0,
    dk ku = D,
    dk lu = 0,
    dl pu = 0,
    dl ku = 0,
    dl lu = D,
    dp pu = D,
    dp ku = 0,
    dp lu = 0 $

```

```

LET
  dk ff(a) =
    2*ku * upaf(1)
    + 2*(ku+pu) * upaf(3)
    + 2*(ku+lu+pu) * upaf(5),
  dl ff(a) =
    2*lu * upaf(2)
    + 2*(lu+pu) * upaf(4)
    + 2*(ku+lu+pu) * upaf(5),
  dp ff(a) =
    2*ku * upaf(3)
    + 2*lu * upaf(4)
    + 2*(ku+lu) * upaf(5)
    + 2*pu * upaf(6) $

t1 := dk (ku * ff(a)) $
t2 := dk (lu * ff(a)) $
t3 := dk (pu * ff(a)) $
t4 := dl (ku * ff(a)) $
t5 := dl (lu * ff(a)) $
t6 := dl (pu * ff(a)) $
t7 := dp (ku * ff(a)) $
t8 := dp (lu * ff(a)) $
t9 := dp (pu * ff(a)) $

FOR ALL ndown, nup LET
  downa (ndown) * upaf (nup) = - ab (ndown, nup) * ff(a)$
FOR ALL n LET
  ab (n,n) * ff(a) = aa (n) * ff(a) $           [ aa(n) is ab(n,n) =  $\alpha_n$ 

FOR ALL a LET
  ff(a) = 1 $                                   [ outlived its usefulness!!

OFF ECHO $
SWITCHOUTPUT (chops) $                         [ write out the recurrence relations
WRITE t1 := t1;
WRITE t2 := t2;
WRITE t3 := t3;
WRITE t4 := t4;
WRITE t5 := t5;
WRITE t6 := t6;
WRITE t7 := t7;
WRITE t8 := t8;
WRITE t9 := t9;

SWITCHOUTPUT NIL $
ON ECHO $

```

Operating on the diagrams with no gluons (diagram g), the $\partial/\partial a_{1,2}$ annihilate the function. Express this, and recalculate the relations.

```

FOR ALL n LET
  ab (n,1) = 0,
  ab (n,2) = 0 $                                [ operating on diagram g

t1 := t1 $
t2 := t2 $
t3 := t3 $
t4 := t4 $
t5 := t5 $
t6 := t6 $
t7 := t7 $
t8 := t8 $
t9 := t9 $

solist := { ab(1,3), ab(1,4), ab(1,5), ab(1,6),
            ab(2,3), ab(2,4), ab(2,5), ab(2,6),
            ab(6,6)} $

OFF ECHO $
REMARK "Solving..." $
SWITCHOUTPUT (chops) $

SOLVE ({t1,t2,t3,t4,t5,t6,t7,t8,t9}, solist);

SHUTOUTPUT (chops) $

END$

```

Appendix D

Lagrangians, Feynman Rules and Dirac algebra

D.1 Lagrangians

The bare Lagrangian for QED and QCD is

$$\mathcal{L} = \bar{\psi}(i\not{D} - m_0)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2a_0}(\partial \cdot A)^2 + \mathcal{L}_g \quad (\text{D.1})$$

The gauge fixing term $\mathcal{L}_{\text{GF}} = -(1/2a_0)(\partial \cdot A)^2$ in both theories modifies the Lagrangian so that it generates an equation of motion which is already in a specific gauge, determined by a_0 .

D.1.1 Quantum Electrodynamics

In QED, the covariant derivative D_μ is obtained from the free-field derivative ∂_μ by the process of minimal coupling, to get $D_\mu = \partial_\mu - ie_0 A_\mu$. The electromagnetic field strength tensor is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu,$$

where A_μ is the photon field. The term \mathcal{L}_g is zero for QED. Note that here, the electron charge is taken to be $-e$, that is, the number e is *positive*.

D.1.2 Quantum Chromodynamics

In QCD, the covariant derivative is $D_\mu = \partial_\mu - ig_0 A_\mu^a T_a$, where A_μ^a are the gluon field operators, the hermitian operators T_a are the generators of $\text{SU}(3)_{\text{couleur}}$, and g_0 is real. The gluon field tensor $F_{\mu\nu}$ is

$$\begin{aligned} F_{\mu\nu} &\equiv \partial_\mu A_\nu - \partial_\nu A_\mu - ig_0[A_\mu, A_\nu] = [D_\mu, D_\nu]/(-ig) \\ &= (\partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g_0 f_{abc} A^{\mu b} A^{\nu c}) T^a, \end{aligned}$$

where f_{abc} are the structure constants of the $\text{SU}(3)$. The ghost term \mathcal{L}_g is

$$\mathcal{L}_g = -\bar{\eta}_a \partial_\mu (\partial^\mu \delta_{ac} - g_0 f_{abc} A_b^\mu) \eta_c,$$

and is required in QCD for a consistent treatment of gluons [10].

D.2 The Feynman rules

The Feynman Rules used are

$$\begin{array}{c} a \\ \hline \xrightarrow{p} \\ \hline b \end{array} = \frac{i\delta^{ab}}{p - m_0}$$

$$\begin{array}{c} a \\ \swarrow \\ \mu, c \\ \nwarrow \\ b \end{array} = ig\gamma_\mu(t^c)_{ab}$$

$$\begin{array}{c} \mu, a \\ \swarrow \\ k \\ \nwarrow \\ \nu, b \end{array} = \frac{-i}{k^2} \left[g_{\mu\nu} + (a_0 - 1) \frac{k_\mu k_\nu}{k^2} \right] \delta^{ab}$$

$$\begin{array}{c} \alpha, a \\ \swarrow \\ p \\ \nwarrow \\ \gamma, c \\ \swarrow \\ q \\ \nwarrow \\ \beta, b \end{array} = -gf^{abc}(g_{\beta\gamma}(q-r)_\alpha + g_{\gamma\alpha}(r-p)_\beta + g_{\alpha\beta}(p-q)_\gamma)$$

$$\begin{array}{c} a \\ \hline \xrightarrow{p} \\ \hline b \end{array} = \frac{i}{p^2} \delta_{ab}$$

$$\begin{array}{c} b \\ \swarrow \\ \mu, a \\ \nwarrow \\ c \end{array} = -gf^{abc}p_\mu \quad \text{where } p \text{ is the momentum of the outgoing positive energy ghost}$$

The hermitian operators t^c are the generators of the symmetry group of the theory.

They are such that

$$\text{Tr } t_a t_b = \frac{1}{2} \delta_{ab}$$

$$\text{Tr } \mathbb{I}_c = N_c$$

$$\text{Tr } t_a = 0$$

$$\delta_a^a = N_c^2 - 1$$

$$t_a t^a = \frac{N_c^2 - 1}{2N_c} \mathbb{I}_c$$

where \mathbb{I}_c is the unit operator in the space of group generators. For QCD, this group is $\text{SU}(N_c)$, and the operators t_a can be represented by the Gell-Mann λ 's: $t^a = \lambda^a/2$.

For QED, we take $N_c = 1$, so that the symmetry group is $U(1)$, the $T^a = \mathbb{I}$, $f_{abc} = 0$, and the coupling $g = e$. These Feynman rules are reproduced in REDUCE form in appendix C, on page 101.

D.3 Dirac algebra

After we write the S-matrix element using the Feynman rules given above, all the manipulations which are done on it, prior to integration in D -dimensional space-time, must also be done in that space. This means we must define the Dirac algebra in D dimensions.

The algebra is

$$\{\gamma_\mu \gamma_\nu\} = 2g_{\mu\nu},$$

where $g_{\mu\nu}$ is the metric tensor in D -dimensional Minkowski space, \mathbb{M}^D , such that $g = \text{diag}(+ - - \dots)$. Thus

$$g_{\mu\nu}g^{\mu\nu} = \delta_\mu^\mu = D.$$

Also

$$\text{Tr}(\text{odd number } \gamma\text{'s}) = 0,$$

$$\text{Tr } \mathbb{I} = f(D),$$

$$\text{Tr } \gamma_\mu \gamma_\nu = f(D)g_{\mu\nu},$$

$$\text{Tr } \gamma_\alpha \gamma_\beta \gamma_\gamma \gamma_\delta = f(D)[g_{\alpha\beta}g_{\gamma\delta} - g_{\alpha\gamma}g_{\beta\delta} + g_{\alpha\delta}g_{\beta\gamma}],$$

where $f(D)$ is a well-behaved function, the form of which is arbitrary except that $f(4) = 4$. We may thus define it to be $f(D) \equiv 4$.

D.4 $SU(N)$

The generators of the group $SU(N)$ are the operators t_a , for $1 \leq a \leq N^2 - 1$, which comprise a Lie algebra,

$$\left. \begin{aligned} [t_a, t_b] &= i f_{abc} t_c \\ \text{Tr } t_a &= 0, \end{aligned} \right\} \quad (\text{D.2})$$

where the *structure constants* $f_{abc} = f_{[abc]} \in \mathbb{R}$ are normalised *via*

$$f_{acd}f_{bcd} = N\delta_{ab}.$$

	definition	SU(N)	SU(3)	U(1)
C_F	$(t_a)_{\alpha\beta}(t_a)_{\beta\gamma} \equiv C_F \delta_{\alpha\gamma}$	$(N^2 - 1)/2N$	$\frac{4}{3}$	1
C_A	$(T_a)_{\alpha\beta}(T_a)_{\beta\gamma} \equiv C_A \delta_{\alpha\gamma}$	N	3	0
T_F	$\text{Tr}(t_a t_b) \equiv T_F \delta_{ab}$	$\frac{1}{2}$	$\frac{1}{2}$	1

Table 9 The factors C_F , C_A and T_F in QCD and QED. The expressions for U(1) are obtained by taking the generator to be $t_1 = \mathbb{I}$, $\lambda_1 = 2\mathbb{I}$, and $f_{abc} = 0 \Rightarrow T_1 = 0$.

We can also define the numbers d_{abc} through the anticommutators $\{t_a, t_b\} = \delta_{ab}\mathbb{I}/N + d_{abc}t_c$. We can also define the *adjoint representation* from the structure constants as the matrices T_a , where

$$(T_a)_{bc} = -if_{abc}.$$

The generators can be represented by

$$t_a = \frac{\lambda_a}{2}, \quad 1 \leq a \leq (N^2 - 1)/N,$$

where the λ_a are hermitian, traceless $N \times N$ matrices. We will also occasionally refer to $\lambda_0 \equiv \mathbb{I}$.

From the above relations, we can define the constants C_F , C_A and T_F , which are summarised in table 9, and which characterise the group.

Specifically, in SU(3), the λ_a are the 3×3 matrices given by Gell-Mann in his original paper [21]:

$$\begin{aligned} \lambda_1 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \lambda_2 &= \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \lambda_3 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ \lambda_4 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} & \lambda_5 &= \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} & \lambda_6 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\ \lambda_7 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} & \lambda_8 &= \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \end{aligned}$$

Appendix E

Integration in $D = 4 - 2\omega$ dimensions

E.1 Mass Integrals — $I(\alpha, \beta; p)$

We define the integral

$$I(\alpha, \beta; p, m_0) \equiv \mu^{2\omega} \int_{\mathbb{M}^D} \frac{d^D k}{(2\pi)^D} \frac{1}{k^{2\alpha} [(p+k)^2 - m_0^2]^\beta}, \quad (\text{E.1})$$

for $\alpha, \beta \in \mathbb{Z}$ and $p \in \mathbb{M}^D$ (we will usually suppress the m_0 argument). To evaluate this, we use Feynman parameters to replace (E.1) by

$$\begin{aligned} I(\alpha, \beta; p) &= \frac{\mu^{2\omega}}{(2\pi)^D} \int_0^1 \frac{(1-x)^{\alpha-1} x^{\beta-1}}{B(\alpha, \beta)} dx \\ &\quad \times \int_{\mathbb{M}^D} \frac{d^D k}{[k^2(1-x) + ((p+k)^2 - m_0^2)x]^{\alpha+\beta}}. \end{aligned} \quad (\text{E.2})$$

Denoting the second integral by \hat{I} , and making the substitution $k \rightarrow \kappa = k + xp$, then Wick rotating $\kappa \rightarrow \hat{\kappa} \in \mathbb{E}^D$, where $\hat{\kappa}_i = \kappa_i$ and $\hat{\kappa}_0 = i\kappa_0$, we have

$$\hat{I} = i(-)^{\alpha+\beta} \int_{\mathbb{E}^D} \frac{d^D \hat{\kappa}}{[\hat{\kappa}^2 + (m_0^2 - p^2)x + p^2 x^2]^{\alpha+\beta}}. \quad (\text{E.3})$$

Having done that, we move to polar coordinates, so that $d^D \hat{\kappa} = dr r^{D-1} d^{D-1} \Omega$, and $\int d^{D-1} \Omega = 2\pi^{D/2}/\Gamma(D/2)$. Now

$$\hat{I} = i(-)^{\alpha+\beta} \frac{2\pi^{D/2}}{\Gamma(D/2)} \int_0^\infty \frac{dr r^{D-1}}{(r^2 + \xi)^{\alpha+\beta}}$$

where $\xi = (m_0^2 - p^2)x + p^2 x^2$. With the pair of substitutions $y = r^2/\xi$ and then $z = 1/(1+y)$, this can be shown to be

$$\hat{I} = i(-)^{\alpha+\beta} \pi^{D/2} [(m_0^2 - p^2)x + p^2 x^2]^{D/2-\alpha-\beta} \frac{\Gamma(\alpha + \beta - D/2)}{\Gamma(\alpha + \beta)}.$$

Replacing this in (E.2), and rearranging

$$I(\alpha, \beta; p) = \frac{i}{(4\pi)^2} (-4\pi\mu^2)^\omega \frac{\Gamma(\alpha + \beta - D/2)}{\Gamma(\alpha)\Gamma(\beta)} (p^2 - m_0^2)^{D/2 - \alpha - \beta} \int_0^1 dx (1-x)^{\alpha-1} x^{D/2 - \alpha - 1} \left[1 - \frac{p^2 x}{p^2 - m_0^2} \right]^{D/2 - \alpha - \beta} \quad (\text{E.4})$$

This integral is of the form of the integral representation of the hypergeometric function

$${}_2F_1(a, b, c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-tz)^{-a} dt$$

with $z = p^2/(p^2 - m_0^2)$, $a = \alpha + \beta - D/2$, $b = D/2 - \alpha$ and $c = D/2$. Substituting this into the above equation, we finally find that

$$I(\alpha, \beta; p) = \frac{i}{(4\pi)^2} (-4\pi\mu^2)^\omega \frac{\Gamma(\alpha + \beta - D/2)\Gamma(D/2 - \alpha)}{\Gamma(\beta)\Gamma(D/2)} (p^2 - m_0^2)^{D/2 - \alpha - \beta} {}_2F_1\left(\alpha + \beta - D/2, D/2 - \alpha, D/2; \frac{p^2}{p^2 - m_0^2}\right). \quad (\text{E.5})$$

We will be concerned with the form of this integral as $\omega \rightarrow 0$ for certain values of α and β . $I(0, 1; p)$ is fairly easy: using (E.5), we have

$$I(0, 1; p) = \frac{i}{(4\pi)^2} (-4\pi\mu^2)^\omega \Gamma(\omega - 1) (p^2 - m_0^2)^{1-\omega} \times {}_2F_1(\omega - 1, 2 - \omega, 2 - \omega; p^2/(p^2 - m_0^2)).$$

But ${}_2F_1(a, b, b; z) = (1 - z)^{-a}$, so that

$$\begin{aligned} I(0, 1; p) &= \frac{i}{(4\pi)^2} \left(4\pi \frac{\mu^2}{m_0^2}\right)^\omega \frac{\Gamma(\omega)}{\omega - 1} (-m_0^2) \\ &= \frac{i}{(4\pi)^2} \frac{m_0^2}{1 - \omega} \left(\frac{1}{\omega} + \ln 4\pi - \gamma_e + \ln \frac{\mu^2}{m_0^2} + O(\omega) \right) \end{aligned}$$

on expansion.

The integral $I(1, 1; p)$ is barely more difficult: using (E.4), we have

$$\begin{aligned} I(1, 1; p) &= \frac{i}{(4\pi)^2} (-4\pi\mu^2)^\omega \Gamma(\omega) \int_0^1 dx (p^2 x(1-x) - m_0^2 x)^{-\omega} \\ &= \frac{i}{(4\pi)^2} \left(1 + \omega \ln \left(4\pi \frac{\mu^2}{m_0^2} \right) + O(\omega^2) \right) \left(\frac{1}{\omega} - \gamma_e + O(\omega) \right) \\ &\quad \int_0^1 dx \left[1 - \omega \left(\ln x + \ln \left(1 - \frac{p^2}{m_0^2} (1-x) \right) + O(\omega^2) \right) \right] \end{aligned}$$

Evaluating the logarithmic integrals, using $\int_0^1 dx \ln(ax+b) = (1+b/a)\ln(a+b) - b/a \ln b - 1$, and expanding, we obtain

$$I(1, 1; p) = \frac{i}{(4\pi)^2} \left[\frac{1}{\omega} + (\ln 4\pi - \gamma_e) + \ln \frac{\mu^2}{m_0^2} + \left(\frac{m_0^2}{p^2} - 1 \right) \ln \left(1 - \frac{p^2}{m_0^2} \right) + 2 + O(\omega) \right]. \quad (\text{E.6})$$

To obtain an expression for $I(2, 1; p)$ near $\omega = 0$, we must work a little harder: we substitute in (E.5), and find that we must evaluate (using the definition of the hypergeometric function [52, eqn 15.1.1])

$$\begin{aligned} \frac{\Gamma(1+\omega)\Gamma(-\omega)}{\Gamma(2-\omega)} {}_2F_1(1+\omega, -\omega, 2-\omega; z) &= \sum_{n=0}^{\infty} \frac{\Gamma(n+1+\omega)\Gamma(n-\omega)}{\Gamma(n+2-\omega)} \frac{z^n}{n!} \\ &= \frac{\Gamma(1+\omega)}{-\omega(1-\omega)} + \sum_{n=1}^{\infty} \frac{\Gamma(n+1+\omega)}{(n+1-\omega)(n-\omega)} \frac{z^n}{n!} \\ &= \frac{\Gamma(1+\omega)}{-\omega(1-\omega)} + 1 + \frac{1-z}{z} \ln(1-z) + O(\omega) \end{aligned}$$

using first the fact that $\Gamma(n+1+\omega)$ is differentiable, and so can be expanded in a Taylor series about $\omega = 0$, and then the identity $-(1/z-1)\ln(1-z) = 1 - \sum_{n=1}^{\infty} z^n/[n(n+1)]$ [53, eqn 1.513.5].

Replacing this in (E.5) and expanding in terms of powers of ω , we finally obtain

$$I(2, 1; p) = -\frac{i}{(4\pi)^2} \frac{1}{p^2 - m_0^2} \times \left[\frac{1}{\omega} + \ln 4\pi - \gamma_e + \ln \frac{\mu^2}{m_0^2} - \left(1 + \frac{m_0^2}{p^2} \right) \ln \left(1 - \frac{p^2}{m_0^2} \right) + O(\omega) \right]. \quad (\text{E.7})$$

We will also use the on-shell limit of (E.5), that is, the limit as $p^2 \rightarrow m_0^2$. Rather than involve oneself in the tricky business of taking the limit of eqn (E.5), it is much simpler to simply substitute $p^2 = m_0^2$ in eqn (E.3) and integrate directly. The result is

$$I(\alpha, \beta; p) = \frac{i}{(4\pi)^2} (-4\pi\mu^2)^\omega (-m_0^2)^{D/2-\alpha-\beta} \times \frac{\Gamma(\alpha+\beta-D/2)\Gamma(D-2\alpha-\beta)}{\Gamma(\beta)\Gamma(D-\alpha-\beta)}. \quad (\text{E.8})$$

Note that $I(\alpha, \beta; p)$ is zero for non-positive integer β , since $\Gamma(z)$ is divergent for non-positive integers z (and ${}_2F_1$ is not).

E.2 Other Integrals

We now define the related, but more complicated integral

$$\tilde{I}(\alpha, \beta; p) \equiv \mu^{2\omega} \int_{\mathbb{M}^D} \frac{d^D k}{(2\pi)^D} \frac{1}{(k^2 - m_0^2)^\alpha [(p+k)^2 - m_0^2]^\beta}. \quad (\text{E.9})$$

The evaluation of this integral proceeds in the same way as the evaluation of $I(\alpha, \beta; p)$ in (E.1) above. We re-express the integral using Feynman parameters, make the change of variables $k \rightarrow \kappa = k + xp$ and Wick rotate to get

$$\tilde{I}(\alpha, \beta; p) = \frac{i\mu^{2\omega}(-)^{\alpha+\beta}}{B(\alpha, \beta)(2\pi)^D} \int_0^1 dx (1-x)^{\alpha-1} x^{\beta-1} \int_{\mathbb{E}^D} \frac{d^D \kappa}{[\kappa^2 + m_0^2 - p^2 x(1-x)]^{\alpha+\beta}}.$$

When we switch to polar coordinates as before, we can perform the integration over \mathbb{E}^D in terms of Γ functions and the denominator in the second integrand, to find that

$$\begin{aligned} \tilde{I}(\alpha, \beta; p) &= \frac{i}{(4\pi)^2} (-4\pi\mu^2)^\omega \frac{\Gamma(\alpha + \beta - D/2)}{\Gamma(\alpha)\Gamma(\beta)} (p^2)^{D/2-\alpha-\beta} \\ &\quad \int_0^1 dx (1-x)^{\alpha-1} x^{\beta-1} \left[x(1-x) - \frac{m_0^2}{p^2} \right]^{D/2-\alpha-\beta}. \end{aligned} \quad (\text{E.10})$$

To progress further, we would have to examine specific values of α and β .

E.3 The Gamma Function $\Gamma(z)$

The Γ function may be defined through the Euler form

$$\Gamma(z) \equiv \int_0^\infty e^{-t} t^{z-1} dt,$$

from which it follows that

$$\Gamma(z+1) = z\Gamma(z).$$

To find an expansion of the Γ function, we can use the *polygamma* function:

$$F^{(n)}(z) \equiv \frac{d^{n+1}}{dz^{n+1}} \ln \Gamma(z+1)$$

$$F^{(n)}(0) = (-)^{n+1} n! \zeta(n+1) \quad , n > 0$$

$$F^{(0)}(0) = -\gamma_e.$$

Using this, and expanding $F^{(0)}(z)$ in a Taylor series,

$$\begin{aligned}\ln \Gamma(z+1) &= \int F^{(0)}(z) dz \\ &= \int dz \left[F^{(0)}(0) + zF^{(1)}(0) + \frac{z^2}{2!}F^{(2)}(0) + \cdots + \frac{z^n}{n!}F^{(n)}(0) \right] \\ &= -\gamma_e z + \zeta(2)\frac{z^2}{2} - \zeta(3)\frac{z^3}{3} + \cdots + (-)^{n+1}\zeta(n+1)\frac{z^{n+1}}{n+1} + \cdots.\end{aligned}$$

Exponentiating, we find

$$z\Gamma(z) = \Gamma(1+z) = 1 - \gamma_e z + \frac{1}{2}(\zeta(2) + \gamma_e^2)z^2 + O(z^3) \quad (\text{E.11})$$

E.4 Notation and Conventions

- Euclidean and Minkowski spaces in D dimensions are denoted \mathbb{E}^D and \mathbb{M}^D respectively.
- The metric in four-dimensional Minkowski space is $g = \text{diag}(+ - - -)$
- The symbols \mathbb{C} , \mathbb{R} and \mathbb{Z} denote the spaces of complex numbers, reals and integers.
- For the list of mass definitions, see table 1 on page 39, and for group theory parameters, table 9 on page 126.
- The charge on the electron is $-e$.
- See also the index of symbols.

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Index of Symbols

In the list below, references are to equations unless otherwise specified

- $\bar{\alpha}$ The running coupling [2.23]
- α_s The strong coupling of QCD
- $\tilde{\alpha}$ An abbreviation for $\tilde{\alpha}(\mu) = \bar{\alpha}(\mu)/\pi d$, and used as an expansion parameter [§3.1]
- β Coupling anomalous dimension. See table 2 [2.28,2.16b]
- γ_m Mass anomalous dimension. See table 2 [2.30,2.16c]
- Λ The RG invariant QCD scale, which appears as a constant of integration in the calculation of the β -function [2.34]
- λ_0 In the context of the Gell-Mann λ 's, the unit matrix \mathbb{I} is sometimes denoted λ_0
- λ_i Gell-Mann's λ -matrices [appx D.4]
- $\Sigma(p)$ The proper self-energy—the sum of all one-particle-irreducible graphs [§2.1.2]
- $\bar{\Sigma}$ The combination $\frac{1}{4} \text{Tr } i(1 + \Delta \not{p})(-i\Sigma)$, used to extract $A_{1,2}$ and $B_{1,2}$ [3.29,§3.2]
- Ω In the context $d^{D-1}\Omega$, this is an angular differential, but in the more common case, $\Omega \equiv g_0^2/(4\pi)^{D/2}p^{2\omega}$, and

it is used as an expansion

parameter [3.9]

$A_{1,2}$ Coefficients of $\alpha^i m_0$ in the fermion self energy [§3.2]

a The gauge parameter

$B_{1,2}$ Coefficients of $\alpha^i(\not{p} - m_0)$ in the fermion self energy [§3.2]

$C_{1,2}$ Coefficients of α^i in Z_m [3.9,3.28]

C_F Colour factor for $SU(N)$.

$C_F = (N^2 - 1)/2N$ [table 9]

C_A Colour factor for $SU(N)$.

$C_A = N$ [table 9]

d Related to the number of fermion

flavours N_F . $d = 12/(33 - 2N_F)$ [2.33]

d_2 Coefficient of α^2 in

$M/m(M)$ [3.8,§3.4]

D Dimension of space-time in dimensional regulation:

$D = 4 - 2\omega$ [§2.1]

Ξ^D Euclidean space in D dimensions

$F_{1,2}$ Coefficients of α^i in Z_2 [4.3]

g Coupling of QCD

$I(\alpha, \beta; p)$ [E.1]

$\tilde{I}(\alpha, \beta; p)$ [E.9]

$m(\mu)$ Renormalised mass, see

Z_m [table 1]

m_τ Before section 2.1.5, the
renormalised mass $m(\mu)$ is
written $m_\tau = m_0/Z_m$ [2.10]

m^{eff} Effective mass [table 1]

\hat{m} The renormalisation-group invariant
mass [table 1]

\bar{m} Average light quark mass:
 $\bar{m} = (m_u + m_d)/2$

M Pole mass. Value of \not{p} at the pole in
the fermion propagator [2.52]

\mathbb{M}^D Minkowski space in D dimensions.
The metric in $D = 4$ dimensions is
 $g = \text{diag}(+ - - -)$

N_c Number of colours in $\text{SU}(N_c)$

N_F Number of quark flavours

N_i [e:A2]

R_i [e:A2]

$r = M_i/M$ Ratio of masses of
intermediate-mass quarks in the
gauge-boson propagator [§ 3.5]

T_F Trace factor. $T_F = \frac{1}{2}$ for
 $\text{SU}(N)$ [table 9]

Z_{11}, Z_{21}, Z_{22} Coefficients in the mass
renormalisation constant Z_m [3.27]

Z_2 Wavefunction renormalisation
constant [2.10]

Z_m Mass renormalisation constant:
 $m_0 = Z_m m(\mu)$ [2.10, 3.27]

Z_M Pole mass renormalisation:
 $m_0 = Z_M M$ [2.52]

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